ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\left(\sqrt{h_a h_b}+\sqrt{h_b h_c}+\sqrt{h_c h_a}\right)^2 \le \frac{(23s^2+12Rr+3r^2)r}{4R}$$

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Solution by Tapas Das-India

$$\sqrt{h_a h_b h_c}=\sqrt{\frac{(abc)^2}{(2R)^2.2R}}=\frac{abc}{2R\sqrt{2R}}=\frac{4Rrs}{2R\sqrt{2R}}=\frac{2rs}{\sqrt{2R}}\ (1)$$

$$\sum h_a h_b=\sum \frac{bc}{2R}\cdot\frac{ca}{2R}=\frac{abc}{4R^2}\sum a=\frac{4Rrs}{4R^2}.2s=\frac{2rs^2}{R}\quad (2)$$

$$\left(\sqrt{h_a h_b}+\sqrt{h_b h_c}+\sqrt{h_c h_a}\right)^2=\sum h_a h_b+2\sqrt{h_a h_b h_c}.\sum\sqrt{h_a}\overset{CBS}{\le}$$

$$\le \sum h_a h_b+2\sqrt{h_a h_b h_c}.\sqrt{3\sum h_a}=$$

$$=\frac{2rs^2}{R}+\frac{4rs}{\sqrt{2R}}\frac{\sqrt{3(s^2+r^2+4Rr)}}{\sqrt{2R}}=\frac{2rs^2}{R}+\frac{4rs}{2R}\sqrt{3(s^2+r^2+4Rr)}$$

$$We\ need\ to\ show:$$

$$\frac{2rs^2}{R}+\frac{4rs}{2R}\sqrt{3(s^2+r^2+4Rr)}\le\frac{(23s^2+12Rr+3r^2)r}{4R}$$

$$\frac{2s}{R}\sqrt{3(s^2+r^2+4Rr)}\le\frac{(23s^2+12Rr+3r^2)}{4R}-\frac{2s^2}{R}$$

$$8s\sqrt{3(s^2+r^2+4Rr)}\le 15s^2+12Rr+3r^2$$

$$\left(8s\sqrt{3(s^2+r^2+4Rr)}\right)^2\le(15s^2+12Rr+3r^2)^2$$

$$192s^2(s^2+r^2+4Rr)\le 225s^4+90s^2r^2+360s^2Rr+144R^2r^2+72Rr^3+9r^4$$

$$s^2(33s^2-102r^2-408Rr)+144R^2r^2+72Rr^3+9r^4\ge 0$$

$$(16Rr-5r^2)(33(16Rr-5r^2)-102r^2-408Rr)+144R^2r^2+72Rr^3+9r^4 \overset{Gerretsen}{\geq} 0$$

$$r^2(2064R^2-4800Rr+1344r^2)\geq 0\ or, 672(R-2r)(3R-r)\geq 0\ true\ by\ Euler$$

Equality holds for an equilateral triangle.