ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$7\left(\frac{1}{w_aw_b}+\frac{1}{w_bw_c}+\frac{1}{w_cw_a}\right)\le\frac{4}{3r^2}+3\left(\frac{1}{m_am_b}+\frac{1}{m_bm_c}+\frac{1}{m_cm_a}\right)$$

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$$7\left(\sum_{cyc}\frac{1}{w_aw_b}\right)-\frac{4}{3r^2}-3\left(\sum_{cyc}\frac{1}{m_am_b}\right)\le$$
$$\frac{7(s^2+2Rr+r^2)}{16Rrs^2}\cdot\frac{9(R+2r)}{4}-\frac{4}{3r^2}-\frac{24}{s^2-4Rr+5r^2}$$
$$\left(\begin{array}{c}\because\sum_{cyc}w_a\le\frac{9(R+2r)}{4}\to\text{Reference : Inequality in Triangle by Dang Ngoc Minh}-166;\\ \text{published at www.ssmrmh.ro and }\sum_{cyc}\frac{1}{m_am_b}\ge\frac{8}{s^2-4Rr+5r^2}\\ \to\text{Reference : Inequality in Triangle by Dang Ngoc Minh}-222;\\ \text{published at www.ssmrmh.ro}\end{array}\right)$$
$$\overset{?}{\le}0\Leftrightarrow(67R-378r)s^4-r(646R^2-5510Rr+2268r^2)s^2+$$
$$r^2(1512R^3+1890R^2r-3213Rr^2-1890r^3)\boxed{\underset{(*)}{\overset{?}{\ge}}}0$$

$\boxed{\text{Case 1}}$ $67R-378r\ge0$ and then, since $P=(67R-378r)(s^2-16Rr+5r^2)^2+$
$2r(749R^2-3628Rr+756r^2)(s^2-16Rr+5r^2)\overset{\text{Gerretsen}}{\ge}0\ \therefore$ to prove $(*)$,
it suffices to prove : LHS of $(*)\overset{?}{\ge}P\Leftrightarrow24Rr^2(R-2r)(347R+102r)\overset{?}{\ge}0$
$\to$ true $\because R\overset{\text{Euler}}{\ge}2r$

$\boxed{\text{Case 2}}$ $67R-378r<0$ and then, since Q
$=(67R-378r)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)\overset{\text{Double-Rouche}}{\ge}0$
$\therefore$ in order to prove $(*)$, it suffices to prove : LHS of $(*)\overset{?}{\ge}Q\Leftrightarrow$
$$(134R^3-409R^2r-1092Rr^2-756r^3)s^2\boxed{\underset{(**)}{\overset{?}{\ge}}}$$
$$r(2144R^4-11244R^3r-9615R^2r^2-628Rr^3+756r^4)$$

$\boxed{\text{Case 2a}}$ $134R^3-409R^2r-1092Rr^2-756r^3\ge0$ and then, LHS of $(**)\overset{\text{Gerretsen}}{\ge}$
$(134R^3-409R^2r-1092Rr^2-756r^3)(16Rr-5r^2)\overset{?}{\ge}$ RHS of $(**)$
$$\Leftrightarrow2015t^3-2906t^2-3004t+1512\overset{?}{\ge}0\left(t=\frac{R}{r}\right)$$
$\Leftrightarrow(t-2)(2015t^2+1124t-756)\overset{?}{\ge}0\to$ true $\because t\overset{\text{Euler}}{\ge}2\Rightarrow(**)$ is true

$\boxed{\text{Case 2b}}$ $134R^3-409R^2r-1092Rr^2-756r^3<0$ and then : LHS of $(**)$

$$\overset{\text{Gerretsen}}{\geq} \left(134R^3 - 409R^2r - 1092Rr^2 - 756r^3\right)\left(4R^2 + 4Rr + 3r^2\right) \overset{?}{\geq} \text{RHS of } (**)$$

$$\Leftrightarrow 268t^5 - 1622t^4 + 2821t^3 + 498t^2 - 2836t - 1512 \overset{?}{\geq} 0$$

$$\Leftrightarrow \left(\frac{t-2}{2}\right)\left((2t-5)^2\left(134t^2 + 127t + 122\right) + 2857t - 1538\right) \overset{?}{\geq} 0 \to \text{true}$$

$\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**)$ is true $\therefore$ combining cases $(2a)$ and $(2b)$, $(**)$ is true whenever $67R - 378r < 0$ and so, combining cases 1 and 2, $(*)$ is true $\forall \Delta ABC$

$$\therefore 7\left(\frac{1}{w_a w_b} + \frac{1}{w_b w_c} + \frac{1}{w_c w_a}\right) \leq \frac{4}{3r^2} + 3\left(\frac{1}{m_a m_b} + \frac{1}{m_b m_c} + \frac{1}{m_c m_a}\right) \forall \Delta ABC,$$

$'' = ''$ iff ΔABC is equilateral (QED)