ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the $fo\ell\ell$owing reℓationship hoℓds :

$$\frac{1}{\sqrt{h_a+h_b+h_c}}+\frac{4}{3}\cdot\sqrt{\frac{2}{R}}\geq\frac{5}{\sqrt{r_a+r_b+r_c}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\frac{1}{\sqrt{h_a+h_b+h_c}}+\frac{4}{3}\cdot\sqrt{\frac{2}{R}}\right)^2=\frac{2R}{s^2+4Rr+r^2}+\frac{32}{9R}+\frac{8}{3}\cdot\sqrt{\frac{2}{R}\cdot\frac{2R}{s^2+4Rr+r^2}}$$

$$\overset{\text{Gerretsen}}{\geq}\frac{2R}{4R^2+8Rr+4r^2}+\frac{32}{9R}+\frac{8}{3}\cdot\sqrt{\frac{2}{R}\cdot\frac{2R}{4R^2+8Rr+4r^2}}$$

$$=\frac{R}{2(R+r)^2}+\frac{32}{9R}+\frac{8}{3(R+r)}=\frac{9R^2+64(R+r)^2+48R(R+r)}{18R(R+r)^2}\overset{?}{\geq}\frac{25}{4R+r}$$

$$\Leftrightarrow 34R^3-75R^2r-18Rr^2+64r^3\overset{?}{\geq}0\Leftrightarrow(R-2r)(34R^2-7Rr-32r^2)\overset{?}{\geq}0$$

$$\rightarrow \text{true} \because R\overset{\text{Euler}}{\geq}2r\therefore\frac{1}{\sqrt{h_a+h_b+h_c}}+\frac{4}{3}\cdot\sqrt{\frac{2}{R}}\geq\frac{5}{\sqrt{r_a+r_b+r_c}}\ \forall\,\Delta\,ABC,$$

$'' = ''$ iff Δ ABC is equiℓater$a\ell$ (QED)