ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$g_a^2g_b^2+g_b^2g_c^2+g_c^2g_a^2\ge 9F^2\ge h_a^2h_b^2+h_b^2h_c^2+h_c^2h_a^2$$

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$$\sum_{cyc} g_a^2g_b^2 \overset{\text{Bogdan Fustei}}{=}$$

$$\sum_{cyc}\left(\left((s-a)^2+\frac{4(s-a)(s-b)(s-c)}{a}\right)\left((s-b)^2+\frac{4(s-a)(s-b)(s-c)}{b}\right)\right)$$

$$=\sum_{cyc}\frac{\left(x^2(y+z)(x+y)+4xyz(x+y)\right)\left(y^2(z+x)+4xyz\right)}{(x+y)(y+z)(z+x)}$$

$$(x=s-a,y=s-b,z=s-c)$$

$$=\frac{\begin{pmatrix}\left(\prod_{cyc}(x+y)\right)\left(\sum_{cyc}x^2y^2\right)+4xyz\sum_{cyc}\left(y^2\left(\sum_{cyc}xy+x^2\right)\right)+\\ 4xyz\sum_{cyc}\left(x^2\left(\sum_{cyc}xy+y^2\right)\right)+16x^2y^2z^2.2\sum_{cyc}x\end{pmatrix}}{(x+y)(y+z)(z+x)}$$

$$=\frac{\left(\prod_{cyc}(x+y)+8xyz\right)\left(\sum_{cyc}x^2y^2\right)+32x^2y^2z^2\sum_{cyc}x+8xyz\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^2\right)}{(x+y)(y+z)(z+x)}$$

$$=\frac{(4Rrs+8r^2s)\left((4Rr+r^2)^2-2r^2s^2\right)+32r^4s^3+8r^2s(4Rr+r^2)(s^2-8Rr-2r^2)}{4Rrs}$$

$$\Rightarrow\sum_{cyc}g_a^2g_b^2=\frac{r^2\left((6R+6r)s^2+16R^3-24R^2r-15Rr^2-2r^3\right)}{R}\overset{?}{\ge}9F^2=9r^2s^2$$

$$\Leftrightarrow 16R^3-24R^2r-15Rr^2-2r^3\underset{(*)}{\overset{?}{\ge}}(3R-6r)s^2$$

$$\text{Now, }(3R-6r)s^2\overset{\text{Gerretsen}}{\le}(3R-6r)(4R^2+4Rr+3r^2)\overset{?}{\le}$$

$$16R^3-24R^2r-15Rr^2-2r^3\Leftrightarrow R^3-3R^2r+4r^3\overset{?}{\ge}0\Leftrightarrow(R-2r)^2(R+r)\overset{?}{\ge}0$$

$$\to\text{true}\Rightarrow(*)\text{ is true}\therefore g_a^2g_b^2+g_b^2g_c^2+g_c^2g_a^2\ge 9F^2\text{ and again},\sum_{cyc}h_a^2h_b^2=$$

$$\sum_{cyc}\frac{b^2c^2.c^2a^2}{16R^4}=\frac{16R^2F^2}{16R^4}.\sum_{cyc}a^2\overset{\text{Leibnitz}}{\le}\frac{9R^2F^2}{R^2}=9F^2$$

$$\Rightarrow 9F^2\ge h_a^2h_b^2+h_b^2h_c^2+h_c^2h_a^2\text{ and so,}$$

$$g_a^2g_b^2+g_b^2g_c^2+g_c^2g_a^2\ge 9F^2\ge h_a^2h_b^2+h_b^2h_c^2+h_c^2h_a^2\ \forall\Delta ABC,$$

$$''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$