ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{1}{w_a^2}+\frac{1}{w_b^2}+\frac{1}{w_c^2}\geq max\left\{\frac{1}{r_ar_b}+\frac{1}{r_br_c}+\frac{1}{r_cr_a},\frac{1}{g_ag_b}+\frac{1}{g_bg_c}+\frac{1}{g_cg_a}\right\}$$

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$$\sum_{cyc}\frac{1}{w_a^2}=\sum_{cyc}\frac{s^2+(s-a)^2+2s(s-a)}{4bcs(s-a)}=\frac{s}{16Rrs}.\sum_{cyc}\frac{a-s+s}{s-a}+$$

$$\frac{1}{16Rrs^2}.\sum_{cyc}a(s-a)+\frac{1}{8Rrs}.\sum_{cyc}a=\frac{s}{16Rrs}.\left(-3+\frac{s(4Rr+r^2)}{r^2s}\right)+\frac{8Rr+2r^2}{16Rrs^2}+$$

$$\frac{1}{4Rr}\therefore\sum_{cyc}\frac{1}{w_a^2}\overset{①}{=}\frac{(2R+r)s^2+r^2(4R+r)}{8Rr^2s^2}\ and\ \sum_{cyc}g_a=\sum_{cyc}\left(\frac{g_a}{\sqrt{h_a}}.\sqrt{h_a}\right)\overset{CBS}{\leq}$$

$$\sqrt{\sum_{cyc}\frac{(s-a)^2+2rh_a}{h_a}}.\sqrt{\frac{s^2+4Rr+r^2}{2R}}(via\ Bogdan\ \text{Fustei})$$

$$=\sqrt{\sum_{cyc}\frac{a(s^2-2sa+a^2)}{2rs}+6r}.\sqrt{\frac{s^2+4Rr+r^2}{2R}}$$

$$=\sqrt{\frac{s^2(2s)-4s(s^2-4Rr-r^2)+2s(s^2-6Rr-3r^2)}{2rs}+6r}.\sqrt{\frac{s^2+4Rr+r^2}{2R}}$$

$$=\sqrt{2R+5r}.\sqrt{\frac{s^2+4Rr+r^2}{2R}}\therefore\left(\sum_{cyc}\frac{1}{g_ag_b}\right)^2=\frac{1}{(g_ag_bg_c)^2}.\left(\sum_{cyc}g_a\right)^2$$

$$\leq\frac{1}{(g_ag_bg_c)^2}.\frac{(2R+5r)(s^2+4Rr+r^2)}{2R}$$

$$=\frac{R}{r^4\left((9R+9r)s^2+(4R+r)^3\right)}.\frac{(2R+5r)(s^2+4Rr+r^2)}{2R}$$

(Reference : Solution to Inequality in Triangle by Mohamed Amine Ben Ajiba – 69; published at www.ssmrmh.ro)

$$\overset{?}{\leq}\left(\sum_{cyc}\frac{1}{w_a^2}\right)^2\overset{via\ ①}{=}\frac{\left((2R+r)s^2+r^2(4R+r)\right)^2}{64R^2r^4s^4}$$

$$\Leftrightarrow-(28R^3+88R^2r-45Rr^2-9r^3)s^6+$$
$$(256R^5+192R^4r-256R^3r^2+192R^2r^3+142Rr^4+19r^5)s^4+$$
$$r^2(1024R^5+1536R^4r+1040R^3r^2+472R^2r^3+117Rr^4+11r^5)s^2+r^4(4R+r)^5$$

$$\boxed{\overset{?}{\underset{(*)}{\geq}}}\ 0\ and\ \because P=-\left(\begin{matrix}28R^3+88R^2r-\\45Rr^2-9r^3\end{matrix}\right)\left(\begin{matrix}s^4-(4R^2+20Rr-2r^2)s^2+\\r(4R+r)^3\end{matrix}\right)s^2$$

$$\overset{\text{Double-Rouche}}{\geq} 0 \therefore \text{in order to prove } (*), \text{it suffices to prove : LHS of } (*) \overset{?}{\geq} P$$

$$\Leftrightarrow (144R^5 - 720R^4r - 1780R^3r^2 + 1304R^2r^3 + 232Rr^4 + r^5)s^4 + r(1792R^6 + 8000R^5r + 3216R^4r^2 - 612R^3r^3 - 412R^2r^4 - 36Rr^5 + 160r^6)s^2 + r^4(4R+r)^5 \boxed{\underset{(**)}{\overset{?}{\geq}}} 0 \text{ and it's trivially true when : coefficient of } s^4 \geq 0 \text{ and when :}$$

$$\text{coefficient of } s^4 < 0, \text{then since } Q = \begin{pmatrix} 144R^5 - 720R^4r - 1780R^3r^2 + \\ 1304R^2r^3 + 232Rr^4 + r^5 \end{pmatrix}(s^4 - (4R^2 + 20Rr - 2r^2)s^2 + r(4R+r)^3)$$

$$\overset{\text{Double-Rouche}}{\geq} 0 \therefore \text{in order to prove } (**), \text{it suffices to prove : LHS of } (**) \overset{?}{\geq} Q \Leftrightarrow$$

$$\begin{pmatrix} T = 144R^6 + 448R^5r - 3452R^4r^2 - 6432R^3r^3 + 7489R^2r^4 + 406Rr^5 - \\ 120r^6 \end{pmatrix} s^2 - r\begin{pmatrix} 2304R^7 - 9792R^6r - 36688R^5r^2 - 2876R^4r^3 + 13520R^3r^4 + \\ 6107R^2r^5 + 994Rr^6 + 56r^7 \end{pmatrix} \boxed{\underset{(***)}{\overset{?}{\geq}}} 0$$

$\boxed{\textit{Case 1}}$ $T \geq 0$ *and then* : LHS of $(***) \overset{\text{Gerretsen}}{\geq} T(16Rr - 5r^2) \overset{?}{\geq}$ RHS of $(***)$

$$\Leftrightarrow 2030t^6 - 2598t^5 - 10347t^4 + 17308t^3 - 4632t^2 - 618t + 68 \overset{?}{\geq} 0$$

$$\left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(2030t^3(t^2-4) + 1462t^4 + 697t^3 + 2462t^2 + 292t - 34) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (***) \text{ is true}$$

$\boxed{\textit{Case 2}}$ $T < 0$; then : LHS of $(***) \overset{\text{Gerretsen}}{\geq} T(4R^2 + 4Rr + 3r^2) \overset{?}{\geq}$ RHS of $(***)$

$$\Leftrightarrow 144t^8 + 16t^7 - 448t^6 - 376t^5 - 813t^4 - 309t^3 + 4376t^2 - 64t - 104 \overset{?}{\geq} 0 \Leftrightarrow$$

$$(t-2)\left((t-2)\begin{pmatrix} 144t^6 + 592t^5 + 1344t^4 + 2632t^3 + \\ 4339t^2 + 6519t + 13096 \end{pmatrix} + 26244\right) \overset{?}{\geq} 0 \rightarrow \text{true}$$

$$\Rightarrow (***) \text{ is true} \therefore \sum_{\text{cyc}} \frac{1}{w_a^2} \geq \sum_{\text{cyc}} \frac{1}{g_a g_b} \text{ and again, } \frac{(2R+r)s^2 + r^2(4R+r)}{8Rr^2s^2} \overset{\text{Gerretsen}}{\geq}$$

$$\frac{(2R+r)(16Rr - 5r^2) + r^2(4R+r)}{8Rr^2s^2} \overset{?}{\geq} \sum_{\text{cyc}} \frac{1}{r_a r_b} = \frac{4R+r}{rs^2} \Leftrightarrow 2r^2(R-2r) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because R \overset{\text{Euler}}{\geq} 2r \therefore \sum_{\text{cyc}} \frac{1}{w_a^2} \geq \sum_{\text{cyc}} \frac{1}{r_a r_b} \text{ and so, } \frac{1}{w_a^2} + \frac{1}{w_b^2} + \frac{1}{w_c^2} \geq$$

$$\max\left\{\frac{1}{r_a r_b} + \frac{1}{r_b r_c} + \frac{1}{r_c r_a}, \frac{1}{g_a g_b} + \frac{1}{g_b g_c} + \frac{1}{g_c g_a}\right\} \forall \Delta ABC,$$

$$" = " \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$