ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{1}{m_b + m_c - m_a} + \frac{1}{m_c + m_a - m_b} + \frac{1}{m_a + m_b - m_c} \geq \frac{1}{r}$$

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$$\text{It is known that : } 4R + r \geq \sum_{cyc} m_a \text{ and } \because \sum_{cyc} \frac{1}{s-a} = \frac{4R+r}{F}$$

$$\therefore 2F.\sum_{cyc} \frac{1}{b+c-a} \geq \sum_{cyc} m_a$$ and implementing it on a triangle with sides $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$ whose medians and area as a consequence of trivial calculations are $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$ and $\frac{F}{3}$ respectively, we arrive at : $\frac{2F}{3}.\sum_{cyc} \frac{1}{\frac{2}{3}(m_b + m_c - m_a)} \geq \frac{1}{2}\sum_{cyc} a$

$$\Rightarrow \sum_{cyc} \frac{1}{m_b + m_c - m_a} \geq \frac{s}{rs} = \frac{1}{r} \text{ and so,}$$

$$\frac{1}{m_b + m_c - m_a} + \frac{1}{m_c + m_a - m_b} + \frac{1}{m_a + m_b - m_c} \geq \frac{1}{r} \; \forall \Delta \text{ ABC},$$

" = " iff Δ ABC is equilateral (QED)