ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$(m_a+m_b+m_c)^3 \geq (r_a+r_b+r_c)(h_a+h_b+h_c)^2$$

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$$\left(\sum_{cyc} m_a\right)^3 \overset{\substack{\text{Chu-Yang}\\ \text{and}\\ \text{Tereshin}}}{\geq} (4s^2-28Rr+29r^2)\left(\sum_{cyc}\frac{b^2+c^2}{4R}\right)$$

$$=\frac{(4s^2-28Rr+29r^2)(s^2-4Rr-r^2)}{R} \overset{?}{\geq} \left(\sum_{cyc} r_a\right)\left(\sum_{cyc} h_a\right)^2$$

$$=\frac{(4R+r)(s^2+4Rr+r^2)^2}{4R^2} \Leftrightarrow (12R-r)s^4-r(208R^2-84Rr+2r^2)s^2+$$

$$r^2(384R^3-400R^2r-128Rr^2-r^3)\ \boxed{\underset{(*)}{\overset{?}{\geq}}}\ 0 \text{ and } \because (12R-r)(s^2-16Rr+5r^2)^2$$

$$\overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (*), \text{it suffices to prove :}$$

$$\text{LHS of } (*) \overset{?}{\geq} (12R-r)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (44R^2-17Rr+2r^2)s^2\ \boxed{\underset{(**)}{\overset{?}{\geq}}}\ r(672R^3-444R^2r+147Rr^2-6r^3) \text{ and again,}$$

$$(44R^2-17Rr+2r^2)s^2 \overset{\text{Gerretsen}}{\geq} (44R^2-17Rr+2r^2)(16Rr-5r^2) \overset{?}{\geq}$$

$$r(672R^3-444R^2r+147Rr^2-6r^3) \Leftrightarrow 16t^3-24t^2-15t-2 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(4t+1)^2 \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore (m_a+m_b+m_c)^3 \geq (r_a+r_b+r_c)(h_a+h_b+h_c)^2\ \forall\, \Delta\, ABC,$$

$$''='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$