ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$7\left(\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\right)\le\frac{5}{r}+2\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{w_a}\overset{CBS}{\le}\sqrt{\sum_{cyc}\frac{1}{h_a}}\cdot\sqrt{\sum_{cyc}\frac{h_a}{w_a^2}}=\sqrt{\frac{1}{2Rr}\cdot\sum_{cyc}\frac{bc}{bc-\frac{a^2bc}{(b+c)^2}}}$$

$$=\sqrt{\frac{1}{2Rr}\cdot\sum_{cyc}\frac{(s+s-a)^2}{4s(s-a)}}=\sqrt{\frac{1}{2Rr}\cdot\left(\frac{s(4Rr+r^2)}{4r^2s}+\frac{s}{4s}+\frac{3}{2}\right)}=\sqrt{\frac{R+2r}{2Rr^2}}$$

$$\therefore 49\left(\sum_{cyc}\frac{1}{w_a}\right)^2\le\frac{49(R+2r)}{2Rr^2}\rightarrow (1)\text{ and }\left(\frac{5}{r}+2\left(\sum_{cyc}\frac{1}{m_a}\right)\right)^2=$$

$$\frac{25}{r^2}+4\left(\sum_{cyc}\frac{1}{m_a}\right)^2+\frac{20}{r}\sum_{cyc}\frac{1}{m_a}\overset{\substack{\text{Dang Ngoc Minh}\\ \text{and}\\ \text{Bergstrom}}}{\ge}\frac{25}{r^2}+\frac{100}{s^2-4Rr+6r^2}+\frac{180}{r\sum_{cyc}m_a}$$

(Reference : Inequality in Triangle by Dang Ngoc Minh – 225; published at www.ssmrmh.ro)

$$\ge\frac{25}{r^2}+\frac{100}{s^2-4Rr+6r^2}+\frac{180}{r(4R+r)}=$$

$$\frac{25(4R+r)(s^2-4Rr+6r^2)+100r^2(4R+r)+180r(s^2-4Rr+6r^2)}{r^2(4R+r)(s^2-4Rr+6r^2)}\overset{?}{\ge}\frac{49(R+2r)}{2Rr^2}$$

$$\Leftrightarrow(4R^2-31Rr-98r^2)s^2\boxed{\underset{(*)}{\overset{?}{\ge}}}r(16R^3-948R^2r-406Rr^2+588r^3)$$

$\boxed{\text{Case 1}}$ $4R^2-31Rr-98r^2\ge 0$ *and then* : LHS of $(*)\overset{\text{Gerretsen}}{\ge}$

$$(4R^2-31Rr-98r^2)(16Rr-5r^2)\overset{?}{\ge}\text{RHS of }(*)$$

$$\Leftrightarrow 48t^3+432t^2-1007t-98\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(48t^2+528t+49)\overset{?}{\ge}0$$

$$\rightarrow\text{true}\because t\overset{\text{Euler}}{\ge}2\Rightarrow(*)\text{ is true}$$

$\boxed{\text{Case 2}}$ $4R^2-31Rr-98r^2<0$ *and then* : LHS of $(*)\overset{\text{Gerretsen}}{\ge}$

$$(4R^2-31Rr-98r^2)(4R^2+4Rr+3r^2)\overset{?}{\ge}\text{RHS of }(*)$$

$$\Leftrightarrow 16t^4-124t^3+444t^2-79t-882\overset{?}{\ge}0$$

$$\Leftrightarrow (t-2)\Big((t-2)\big((t-2)(16t-28)+84\big)+721\Big) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (*) \text{ is true } \therefore \text{ combining both cases, } (*) \text{ is true } \forall\, \Delta\, ABC$$

$$\therefore \left(\frac{5}{r}+2\left(\sum_{cyc}\frac{1}{m_a}\right)\right)^2 \geq \frac{49(R+2r)}{2Rr^2} \overset{\text{via } (*)}{\geq} 49\left(\sum_{cyc}\frac{1}{w_a}\right)^2 \text{ and so,}$$

$$7\left(\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\right) \leq \frac{5}{r}+2\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right) \forall\, \Delta\, ABC,$$

$$'' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$