ROMANIAN MATHEMATICAL MAGAZINE

In $any\ \Delta$ ABC the $following$ $relationship$ $holds$:

$$\frac{2a}{b+c} \leq 1+\sqrt{1-\frac{2r}{R}}$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Let } s=\sin\frac{A}{2} \text{ and } c=\cos\frac{B-C}{2} \text{ and then}: \frac{2a}{b+c} \overset{?}{\leq} 1+\sqrt{1-\frac{2r}{R}}$$

$$\Leftrightarrow \frac{8R\sin\frac{A}{2}\cos\frac{A}{2}}{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}-1 \overset{?}{\leq} \sqrt{\frac{R(1-4sc+4s^2)}{R}} \Leftrightarrow \frac{2s-c}{c} \underset{(*)}{\overset{?}{\leq}} \sqrt{1-4sc+4s^2} \text{ and it's's}$$

$$\text{trivially true when}: 2s \leq c \text{ and when}: 2s > c, \text{then } (*) \Leftrightarrow$$

$$\frac{(2s-c)^2}{c^2} \overset{?}{\leq} 1-4sc+4s^2 \Leftrightarrow c^2-4sc^3+4s^2c^2 \overset{?}{\geq} 4s^2-4sc+c^2$$

$$\Leftrightarrow c-s-c^2(c-s) \overset{?}{\geq} 0 \Leftrightarrow (c-s)(1-c^2) \overset{?}{\geq} 0 \to \text{true}$$

$$\because c=\cos\frac{B-C}{2}=\frac{b+c}{a}.\sin\frac{A}{2} > s \text{ and } 1 \geq \cos^2\frac{B-C}{2}=c^2 \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{2a}{b+c} \leq 1+\sqrt{1-\frac{2r}{R}}\ \forall\ \Delta\ ABC, ''='' \text{ iff } \Delta\ ABC \text{ is isosceles with } b=c\ (QED)$$

Solution 2 by Tapas Das-India

$$Let\ a=y+z, b=z+x, c=x+y\ then:$$

$$2s=a+b+c=2(x+y+z)\ or\ s=x+y+z$$

$$F=\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{xyz(x+y+z)}$$

$$\frac{r}{R}=\frac{\frac{F}{s}}{\frac{abc}{4F}}=\frac{4F^2}{sabc}=\frac{4xyz}{(x+y)(y+z)(z+x)}$$

$$\frac{2a}{b+c}-1=\frac{2(y+z)}{2x+y+z}-1=\frac{y+z-2x}{2x+y+z}$$

We need to show:

$$\frac{2a}{b+c} \le 1 + \sqrt{1 - \frac{2r}{R}} \ \ or \ \ \frac{2a}{b+c} - 1 \le \sqrt{1 - \frac{2r}{R}}$$

$$\frac{y+z-2x}{2x+y+z} \le \sqrt{1 - \frac{4xyz}{(x+y)(y+z)(z+x)}}$$

$$\left(\frac{y+z-2x}{2x+y+z}\right)^2 - 1 \overset{squaring}{\le} \frac{4xyz}{(x+y)(y+z)(z+x)}$$

$$\frac{(y+z-2x)^2 - (2x+y+z)^2}{(2x+y+z)^2} \le \frac{4xyz}{(x+y)(y+z)(z+x)}$$

$$\frac{8x(y+z)}{(2x+y+z)^2} \le \frac{4xyz}{(x+y)(y+z)(z+x)}$$

$$(y+z)^2(x+y)(z+x) \ge yz(2x+y+z)^2$$

$$L.H.S = (y+z)^2(x+y)(z+x) = (y+z)^2(x^2 + x(y+z) + yz) =$$
$$= (y+z)^2x^2 + x(y+z)^3 + yz(y+z)^2$$
$$R.H.S = yz(2x+y+z)^2 = yz\left(4x^2 + \acute{y}z(y+z)^2 + 4x(y+z)\right) =$$
$$= 4x^2yz + yz(y+z)^2 + 4xyz(y+z)$$

$$L.H.S - R.H.S = (y+z)^2x^2 + x(y+z)^3 - 4x^2yz - 4xyz(y+z) =$$
$$= x^2((y+z)^2 - 4yz) + x(y+z)((y+z)^2 - 4yz) =$$
$$= x^2(y-z)^2 + x(y+z)(y-z)^2 = (y-z)^2x(x+y+z) \ge 0.$$

$$So\ L.H.S - R.H.S \ge 0\ or\ L.H.S \ge R.H.S$$

$$(y+z)^2(x+y)(z+x) \ge yz(2x+y+z)^2$$

$$Equality\ holds\ for\ y = z\ or\ b = c.$$