ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$5w_a w_b w_c - 4r_a r_b r_c \le g_a g_b g_c \le \frac{2}{3} w_a w_b w_c + \frac{1}{3} h_a h_b h_c$$

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$$5w_a w_b w_c - 4r_a r_b r_c = 4rs^2\left(\frac{20Rr}{s^2+2Rr+r^2} - 1\right)$$

$$= 4rs^2\left(\frac{2Rr+4r^2-(s^2-16Rr+5r^2)}{s^2+2Rr+r^2}\right) \overset{?}{\le} g_a g_b g_c \Leftrightarrow \frac{16r^2s^4(2Rr+4r^2)^2}{(s^2+2Rr+r^2)^2} \overset{?}{\le}$$

$$(g_a g_b g_c)^2 + \frac{16r^2s^4(s^2-16Rr+5r^2)^2}{(s^2+2Rr+r^2)^2} + \frac{8rs^2(s^2-16Rr+5r^2)}{s^2+2Rr+r^2}.g_a g_b g_c \text{ and } \because$$

$$g_a g_b g_c \ge h_a h_b h_c \therefore \text{it suffices to prove : } \frac{16r^2s^4(2Rr+4r^2)^2}{(s^2+2Rr+r^2)^2} \overset{?}{\le}$$

$$r^4.\frac{(9R+9r)s^2+(4R+r)^3}{R} + \frac{16r^2s^4(s^2-16Rr+5r^2)^2}{(s^2+2Rr+r^2)^2} +$$

$$\frac{8rs^2(s^2-16Rr+5r^2)}{s^2+2Rr+r^2}.\frac{2r^2s^2}{R}$$

(Reference : Solution to Inequality in Triangle by Mohamed Amine Ben Ajiba – 69; published at www.ssmrmh.ro)

$$\Leftrightarrow (16R+16r)s^8 - r(512R^2+55Rr-105r^2)s^6 +$$

$$r^2(4096R^3-3244R^2r+114Rr^2+99r^3)s^4 +$$

$$r^3(256R^4+356R^3r+216R^2r^2+73Rr^3+11r^4)s^2 +$$

$$r^4(256R^5+448R^4r+304R^3r^2+100R^2r^3+16Rr^4+r^5) \boxed{\underset{(*)}{\overset{?}{\ge}}} 0 \text{ and } \because P =$$

$$(16R+16r)(s^2-16Rr+5r^2)^4 + r(512R^2+649Rr-215r^2)(s^2-16Rr+5r^2)^3$$

$$+r^2(4096R^3+11012R^2r-6981Rr^2+924r^3)(s^2-16Rr+5r^2)^2 +$$

$$4r^3(64R^4+18873R^3r-16004R^2r^2+4094Rr^3-276r^4)(s^2-16Rr+5r^2)$$

$$\overset{\text{Gerretsen}}{\ge} 0 \therefore \text{in order to prove } (*), \text{it suffices to prove : LHS of } (*) \overset{?}{\ge} P$$

$$\Leftrightarrow 1088t^5 - 576t^4 - 4209t^3 + 1848t^2 + 428t - 176 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(1088t^4+1600t^3-1009t^2-170t+88) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$$\Rightarrow (*) \text{ is true} \therefore 5w_a w_b w_c - 4r_a r_b r_c \le g_a g_b g_c \text{ and again,}$$

$$g_a g_b g_c \overset{?}{\le} \frac{2}{3} w_a w_b w_c + \frac{1}{3} h_a h_b h_c \Leftrightarrow 9r^4.\frac{(9R+9r)s^2+(4R+r)^3}{R} \overset{?}{\le}$$

$$4.\frac{256R^2r^4s^4}{(s^2+2Rr+r^2)^2} + \frac{4r^4s^4}{R^2} + 4.\frac{16Rr^2s^2}{s^2+2Rr+r^2}.\frac{2r^2s^2}{R}$$

$$\Leftrightarrow 4s^8 + (47R^2-65Rr+8r^2)s^6 +$$

$$(448R^4-500R^3r-450R^2r^2-155Rr^3+4r^4)s^4 -$$

$$r(2304R^5+3204R^4r+1944R^3r^2+657R^2r^3+99Rr^4)s^2$$

$$-r^2\left(2304R^6+4032R^5r+2736R^4r^2+900R^3r^3+144R^2r^4+9Rr^5\right)\boxed{\overset{?}{\underset{(**)}{\geq}}}0 \text{ and } \because$$

$$Q=4(s^2-16Rr+5r^2)^4+r(47R^2+191Rr-72r^2)(s^2-16Rr+5r^2)^3+$$
$$(448R^4+1756R^3r+1869R^2r^2-2636Rr^3+484r^4)(s^2-16Rr+5r^2)^2+$$
$$4r\begin{pmatrix}3008R^5+3103R^4r-4572R^3r^2-\\5422R^2r^3+3016Rr^4-360r^5\end{pmatrix}(s^2-16Rr+5r^2)\overset{\text{Gerretsen}}{\geq}0$$

$$\therefore \text{ in order to prove } (**), \text{it suffices to prove : LHS of } (**)\overset{?}{\geq}Q$$

$$\Leftrightarrow 18880t^6-12736t^5-56599t^4+7304t^3+13784t^2-4576t+400\overset{?}{\geq}0$$

$$\Leftrightarrow (t-2)\left(18880t^5+25024t^4-6551t^3-5798t^2+2188t-200\right)\overset{?}{\geq}0\to \text{true}$$

$$\because t\overset{\text{Euler}}{\geq}2\Rightarrow(**)\text{ is true}\therefore g_ag_bg_c\leq\frac{2}{3}w_aw_bw_c+\frac{1}{3}h_ah_bh_c \text{ and so,}$$

$$5w_aw_bw_c-4r_ar_br_c\leq g_ag_bg_c\leq\frac{2}{3}w_aw_bw_c+\frac{1}{3}h_ah_bh_c\ \forall\ ABC,$$
$$''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$