ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$2n_a \ge \sqrt{\frac{w_a}{g_a}}\cdot\left(p_a + m_a\cdot\sqrt{\frac{w_a}{g_a}}\right)$$

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$$m_a n_a \overset{?}{\ge} p_a^2 + \frac{(b-c)^2}{18} \overset{\substack{\text{Fustei}\\ \text{and}\\ \text{Ajiba}}}{\Leftrightarrow}$$

$$\left(s(s-a)+\frac{(b-c)^2}{4}\right)\left(s(s-a)+\frac{s(b-c)^2}{a}\right) \overset{?}{\ge} \left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2$$

$$+\frac{(b-c)^4}{324}+\frac{(b-c)^2}{9}.\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a}+\frac{1}{4}\right)+\frac{s(b-c)^4}{4a} \overset{?}{\ge} \frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4}+$$

$$2s(s-a).\frac{s(3s+a)(b-c)^2}{(2s+a)^2}+\frac{(b-c)^4}{324}+s(s-a).\frac{(b-c)^2}{9}+\frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a}+\frac{1}{4}-\frac{2s(3s+a)(b-c)^2}{(2s+a)^2}-\frac{1}{9}\right)+$$

$$\left(\frac{s}{4a}-\frac{s^2(3s+a)^2}{(2s+a)^4}-\frac{1}{324}-\frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2 \overset{?}{\ge} 0(\because (b-c)^2 \ge 0)$$

$$(\because (b-c)^2 \ge 0) \Leftrightarrow \frac{s(s-a)(144s^3-52s^2a-16sa^2+5a^3)}{36a(2s+a)^2}+$$

$$\frac{1296s^5-772s^4a-608s^3a^2+48s^2a^3+37sa^4-a^5}{324a(2s+a)^4}.(b-c)^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow \frac{s(s-a)\left((s-a)(144s^2+92sa+76a^2)+81a^3\right)}{36a(2s+a)^2}+$$

$$\frac{(s-a)\left((s-a)(1296s^3+1820s^2a+1736sa^2+1700a^3)+1701a^4\right)}{324a(2s+a)^4}.(b-c)^2$$

$$\overset{?}{\ge} 0 \to \text{true (strict inequality)} \therefore m_a n_a \ge p_a^2+\frac{(b-c)^2}{18} \ge p_a^2 \Rightarrow m_a n_a \ge p_a^2 \to (1)$$

$$\text{and } n_a g_a \overset{\text{Bogdan Fustei}}{\ge} m_a w_a \to (2) \therefore (1)\bullet(2) \Rightarrow (m_a n_a)(n_a g_a) \ge p_a^2.m_a w_a$$

$$\Rightarrow n_a \ge \sqrt{\frac{w_a}{g_a}}.p_a \text{ and } (2) \Rightarrow n_a \ge \sqrt{\frac{w_a}{g_a}}.m_a.\sqrt{\frac{w_a}{g_a}}$$

$$\therefore\ 2\mathbf{n}_a \ge \sqrt{\frac{\mathbf{w}_a}{g_a}}.\mathbf{p}_a + \sqrt{\frac{\mathbf{w}_a}{g_a}}.\mathbf{m}_a.\sqrt{\frac{\mathbf{w}_a}{g_a}}\ \textit{a}\mathbf{nd\ so}, 2\mathbf{n}_a \ge \sqrt{\frac{\mathbf{w}_a}{g_a}}.\left(\mathbf{p}_a + \mathbf{m}_a.\sqrt{\frac{\mathbf{w}_a}{g_a}}\right)$$

$$\forall\,\Delta\,\mathbf{ABC}, ''='' \ \mathbf{iff\ b = c}\ (\mathbf{QED})$$