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$$\text{If } I \to \text{incenter in any } \Delta ABC \text{ and } I(x,y)=\frac{1}{e}\left(\frac{y^y}{x^x}\right)^{\frac{1}{y-x}}, 0<x<y, \text{then :}$$

$$\sum_{cyc} I\left(\sqrt{\frac{r_a}{h_a}},\sqrt{\frac{r_b}{h_b}}\right)<\sqrt{\frac{2R-r+AI+BI+CI}{r}}$$

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$$\text{For } 0<x<y, \text{ we have : } I(x,y)=\frac{1}{e}\left(\frac{y^y}{x^x}\right)^{\frac{1}{y-x}}<\frac{x+y}{2}$$

Reference :

(Article titled "Geometric Inequalities with Classical Means" by Bogdan Fustei, published at : www.ssmrmh.ro)

$$\therefore \sum_{cyc} I\left(\sqrt{\frac{r_a}{h_a}},\sqrt{\frac{r_b}{h_b}}\right)<\sum_{cyc}\frac{\sqrt{\frac{r_a}{h_a}}+\sqrt{\frac{r_b}{h_b}}}{2}=\sum_{cyc}\sqrt{\frac{s\tan\frac{A}{2}.4R\cos\frac{A}{2}\sin\frac{A}{2}}{2rs}}$$

$$=\sqrt{\frac{2R}{r}}.\sum_{cyc}\sin\frac{A}{2}\Rightarrow\left(\sum_{cyc} I\left(\sqrt{\frac{r_a}{h_a}},\sqrt{\frac{r_b}{h_b}}\right)\right)^2<\frac{2R}{r}.\left(\sum_{cyc}\sin^2\frac{A}{2}+2\sum_{cyc}\sin\frac{A}{2}\sin\frac{B}{2}\right)$$

$$=\frac{2R}{r}.\frac{2R-r}{2R}+\frac{4R}{r}.\sum_{cyc}\sin\frac{A}{2}\sin\frac{B}{2}\overset{?}{=}\frac{2R-r+AI+BI+CI}{r}$$

$$\Leftrightarrow 16R^2\left(\sum_{cyc}\sin^2\frac{A}{2}\sin^2\frac{B}{2}+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.\sum_{cyc}\sin\frac{A}{2}\right)\overset{?}{=}\sum_{cyc}AI^2+2\sum_{cyc}AI.BI$$

$$\Leftrightarrow 16R^2.\frac{r^2}{16R^2}\sum_{cyc}\csc^2\frac{A}{2}+16R^2.\frac{r}{2R}.\sum_{cyc}\sin\frac{A}{2}\overset{?}{=}$$

$$r^2.\sum_{cyc}\csc^2\frac{A}{2}+\frac{2r^2}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}.\sum_{cyc}\sin\frac{A}{2}\Leftrightarrow 8Rr.\sum_{cyc}\sin\frac{A}{2}\overset{?}{=}\frac{2r^2.4R}{r}.\sum_{cyc}\sin\frac{A}{2}$$

$$\to \text{true} \therefore \sum_{cyc} I\left(\sqrt{\frac{r_a}{h_a}},\sqrt{\frac{r_b}{h_b}}\right)<\sqrt{\frac{2R-r+AI+BI+CI}{r}}\ \forall\Delta ABC\ (QED)$$