ROMANIAN MATHEMATICAL MAGAZINE

If in $\Delta ABC, AB = c, BC = a, CA = b, x = C, m(\sphericalangle BAC) > 90°$,

$tanx = \frac{c}{a} = \frac{b}{d}, b^2 + d^2 = a^2$ then find: $tan2x$.

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Cosinus theorem for triangle ABC: $c^2 = a^2 + b^2 - 2ab\cos x$ $and\ tanx = \frac{c}{a} = \frac{b}{d} \to d = \frac{ab}{c}$

$$b^2 + d^2 = a^2 \to \quad b^2 + \left(\frac{ab}{c}\right)^2 = a^2 \to b^2c^2 + a^2b^2 = a^2c^2 \to \left(\frac{c}{a}\right)^2 + 1 = \left(\frac{c}{b}\right)^2 \to$$

$$1 + tan^2x = \left(\frac{c}{b}\right)^2 \to \frac{1}{cos^2x} = \left(\frac{c}{b}\right)^2 \to cosx = \frac{b}{c} \qquad c^2 = a^2 + b^2 - 2ab\frac{b}{c} \to$$

$$c^3 = a^2c + b^2c - 2ab^2 \to 2ab^2 - b^2c = a^2c - c^3 \to 2ab^2 - b^2c = c(a^2 - c^2)$$

on the other hand $tan2x = \frac{2tanx}{1-tan^2x} = \frac{2\frac{c}{a}}{1-\left(\frac{c}{a}\right)^2} = \frac{2ac}{a^2-c^2} \to$

$$a^2 - c^2 = \frac{2ac}{tan2x}.\ \ 2ab^2 - b^2c = c\frac{2ac}{tan2x} \to \frac{2ab^2}{ac^2} - \frac{b^2c}{ac^2} = \frac{2}{tan2x} \to$$

$$2\left(\frac{b}{c}\right)^2 - \left(\frac{b}{c}\right)^2\frac{c}{a} = \frac{2}{tan2x} \to 2cos^2x - tanxcos^2x = \frac{2}{tan2x}$$

Remark:

$$cos^2x = \frac{1}{1+tan^2x}\ and\ tan2x = \frac{2tanx}{1-tan^2x} \to \frac{2}{1+tan^2x} - \frac{tanx}{1+tan^2x} = \frac{1-tan^2x}{tanx} \to$$

$$\frac{2 - tanx}{1 + tan^2x} = \frac{1 - tan^2x}{tanx} \to (1 - tan^2x)(1 + tan^2x) = tanx(2 - tanx)$$

$$\to tan^4x - tan^2x + 2tanx - 1 = 0 \to (tan^2x)^2 - (tanx - 1)^2 = 0$$

$$\to \begin{cases} tan^2x - tanx + 1 = 0 \quad D < 0 \\ tan^2x + tanx - 1 = 0 \to \\ tanx = \dfrac{-1 + \sqrt{5}}{2} \end{cases}$$

$$tan2x = 2$$