ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0,n\in\mathbb{N}$ and $abc=1$ then :

$$\sum_{cyc}\frac{\sqrt{a(b^4+c^4)}+\sqrt{c(c^4+a^4)}}{\left(\frac{b}{a}\right)^n.\sqrt{ab}+\left(\frac{c}{a}\right)^n.\sqrt{bc}}\ge 3\sqrt{2}$$

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$$\text{Via Walter Janous},\forall A',B',C',x',y',z'>0,$$

$$\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')\ge\sqrt{3\sum_{cyc}A'B'}\ (\text{via})\to ①$$

$$\sum_{cyc}\frac{\sqrt{a(b^4+c^4)}+\sqrt{c(c^4+a^4)}}{\left(\frac{b}{a}\right)^n.\sqrt{ab}+\left(\frac{c}{a}\right)^n.\sqrt{bc}}=\sum_{cyc}\frac{\sqrt{\frac{b^4+c^4}{c}}+\sqrt{\frac{c^4+a^4}{a}}}{\left(\frac{b}{a}\right)^n.\sqrt{\frac{b}{c}}+\left(\frac{c}{a}\right)^n.\sqrt{\frac{b}{a}}}=$$

$$=\sum_{cyc}\frac{\frac{a^n}{\sqrt{b}}.\left(\sqrt{\frac{b^4+c^4}{c}}+\sqrt{\frac{c^4+a^4}{a}}\right)}{\frac{b^n}{\sqrt{c}}+\frac{c^n}{\sqrt{a}}}=$$

$$=\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')$$

$$\left(x'=\frac{a^n}{\sqrt{b}},y'=\frac{b^n}{\sqrt{c}},z'=\frac{c^n}{\sqrt{a}},A'=\sqrt{\frac{a^4+b^4}{b}},B'=\sqrt{\frac{b^4+c^4}{c}},C'=\sqrt{\frac{c^4+a^4}{a}}\right)\overset{\text{via } ①}{\ge}$$

$$\sqrt{3\sum_{cyc}\left(\sqrt{\frac{a^4+b^4}{b}}.\sqrt{\frac{b^4+c^4}{c}}\right)}\overset{AM-GM}{\ge}3.\sqrt{\sqrt[3]{\frac{(a^4+b^4)(b^4+c^4)(c^4+a^4)}{abc}}}\overset{Cesaro}{\ge}3.\sqrt[6]{\frac{8a^4b^4c^4}{abc}}$$

$$\overset{abc=1}{=}3.\sqrt[6]{8}=3\sqrt{2}\text{ and so},\sum_{cyc}\frac{\sqrt{a(b^4+c^4)}+\sqrt{c(c^4+a^4)}}{\left(\frac{b}{a}\right)^n.\sqrt{ab}+\left(\frac{c}{a}\right)^n.\sqrt{bc}}\ge 3\sqrt{2}$$

$$\forall a,b,c>0\ \middle|\ abc=1\wedge n\in\mathbb{N},''=''\text{ iff } a=b=c=1\ (\text{QED})$$