ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $abc = 1$ then :

$$\sum_{cyc} \frac{b(ab)^{2026} + c.c^{2.2026}}{(ab)^{2026} + (bc)^{2026}} \geq 3$$

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$$\forall A', B', C', x', y', z' > 0,$$

$$\frac{x'}{y'+z'}(B'+C') + \frac{y'}{z'+x'}(C'+A') + \frac{z'}{x'+y'}(A'+B') \underbrace{\overset{\text{Walter Janous}}{\geq}}_{①} \sqrt{3\sum_{cyc} A'B'}$$

$$\text{Now, } \sum_{cyc} \frac{b(ab)^{2026} + c.c^{2.2026}}{(ab)^{2026} + (bc)^{2026}} = \sum_{cyc} \frac{\frac{b^{n+1}}{b^n c^n} + c^{2n+1}}{\frac{b^n}{b^n c^n} + (bc)^n}$$

$$\left(n = 2026 \text{ and } \because a^n = \frac{1}{b^n c^n} \text{ and analogs}\right) = \sum_{cyc} \frac{\frac{b}{c^n} + c^{2n+1}}{\frac{1}{c^n} + \frac{1}{a^n}}$$

$$\left(\because (bc)^n = \frac{1}{a^n} \text{ and analogs}\right) = \sum_{cyc} \frac{\frac{1}{b^n}\left(\frac{b^{n+1}}{c^n} + b^n.c^{2n+1}\right)}{\frac{1}{c^n} + \frac{1}{a^n}} = \sum_{cyc} \frac{\frac{1}{b^n}\left(\frac{b^{n+1}}{c^n} + \frac{c^{n+1}}{a^n}\right)}{\frac{1}{c^n} + \frac{1}{a^n}}$$

$$\left((bc)^n = \frac{1}{a^n} \text{ \& analogs}\right) = \frac{x'}{y'+z'}(B'+C') + \frac{y'}{z'+x'}(C'+A') + \frac{z'}{x'+y'}(A'+B')$$

$$\left(x' = \frac{1}{b^n}, y' = \frac{1}{c^n}, z' = \frac{1}{a^n}, A' = \frac{a^{n+1}}{b^n}, B' = \frac{b^{n+1}}{c^n}, C' = \frac{c^{n+1}}{a^n}\right)$$

$$\overset{\text{via } ①}{\geq} \sqrt{3\sum_{cyc}\left(\frac{a^{n+1}}{b^n}.\frac{b^{n+1}}{c^n}\right)} \overset{\text{AM-GM}}{\geq} 3.\sqrt{\sqrt[3]{\left(\frac{a^{n+1}.b^{n+1}.c^{n+1}}{a^n b^n c^n}\right)^2}} = 3.\sqrt[3]{abc} = 3$$

$$\therefore \sum_{cyc} \frac{b(ab)^{2026} + c.c^{2.2026}}{(ab)^{2026} + (bc)^{2026}} \geq 3 \ \forall a, b, c > 0 \mid abc = 1,$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$