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If $a,b,c,d>0$, *then prove that*:

$$\frac{abc+bcd+cda+dab}{a+b+c+d}\le$$

$$\le\frac{1}{4}\cdot\sqrt[3]{(a+b)(a+c)(a+d)(b+c)(b+d)(c+d)}\le\frac{(a+b+c+d)^2}{16}$$

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$$3(a+b+c+d)=$$

$$=(a+b)+(a+c)+(a+d)+(b+c)+(b+d)+(c+d)\overset{\text{AM-GM}}{\ge}$$

$$6\cdot\sqrt[6]{(a+b)(a+c)(a+d)(b+c)(b+d)(c+d)}$$

$$\Rightarrow(a+b+c+d)\cdot\frac{1}{4}\cdot\sqrt[3]{(a+b)(a+c)(a+d)(b+c)(b+d)(c+d)}\ge$$

$$\frac{1}{2}\cdot\sqrt{(a+b)(a+c)(a+d)(b+c)(b+d)(c+d)}$$

$$=\frac{1}{2}\cdot\sqrt{\big((a+b)(b+c)(c+d)\big)\cdot\big((a+c)(a+d)(b+d)\big)}$$

$$=\frac{\left(\sqrt{\begin{matrix}(abc+bcd+cda+dab)\\+\big(c^2(a+b)+b^2(c+d)\big)\end{matrix}}\right)\cdot\left(\sqrt{\begin{matrix}(abc+bcd+cda+dab)\\+\big(a^2(b+d)+d^2(a+c)\big)\end{matrix}}\right)}{2}$$

$$\overset{\text{Reverse CBS}}{\ge}\frac{abc+bcd+cda+dab+\sqrt{\big(c^2(a+b)+b^2(c+d)\big)\cdot\big(a^2(b+d)+d^2(a+c)\big)}}{2}$$

$$\overset{?}{\ge}abc+bcd+cda+dab$$

$$\Leftrightarrow\sqrt{\big(c^2(a+b)+b^2(c+d)\big)\cdot\big(a^2(b+d)+d^2(a+c)\big)}\ \boxed{\underset{(*)}{\overset{?}{\ge}}}\ abc+bcd+cda+dab$$

$\boxed{\text{Case 1}}$ $ac+bd\ge ab+cd$ *and then*: LHS of $(*)\overset{\text{Reverse CBS}}{\ge}$

$$ca\cdot\sqrt{(a+b)(b+d)}+bd\cdot\sqrt{(c+d)(a+c)}$$

$$=ca\cdot\sqrt{(b+a)(b+d)}+bd\cdot\sqrt{(c+d)(c+a)}\overset{\text{Reverse CBS}}{\ge}$$

$$ca\cdot\big(b+\sqrt{ad}\big)+bd\cdot\big(c+\sqrt{ad}\big)\overset{?}{\ge}abc+bcd+cda+dab$$

$$\Leftrightarrow\sqrt{ad}\cdot(ca+bd)\overset{?}{\ge}ad(b+c)\Leftrightarrow ca+bd\underset{(1)}{\overset{?}{\ge}}\sqrt{ad}\cdot(b+c)\ \text{and}\ \because\sqrt{ad}\overset{\text{AM-GM}}{\le}\frac{a+d}{2}$$

$\therefore$ in order to prove (1), it suffices to prove: $2(ac+bd)\overset{?}{\ge}(a+d)(b+c)$

$$=ab+ac+bd+cd\Leftrightarrow ac+bd\overset{?}{\ge}ab+cd\to\text{true}\Rightarrow(1)\Rightarrow(*)\ \text{is true}$$

$\boxed{\text{Case 2}}$ $ab+cd\ge ac+bd$ *and then*: LHS of $(*)\overset{\text{Reverse CBS}}{\ge}$

$$cd\cdot\sqrt{(a+b)(a+c)}+ab\cdot\sqrt{(b+d)(c+d)}$$

$$=cd\cdot\sqrt{(a+b)(a+c)}+ab\cdot\sqrt{(d+b)(d+c)}\overset{\text{Reverse CBS}}{\ge}$$

$$\mathrm{cd}.\left(a+\sqrt{\mathrm{bc}}\right)+a\mathrm{b}.\left(\mathrm{d}+\sqrt{\mathrm{bc}}\right)\overset{?}{\geq}a\mathrm{bc}+\mathrm{bcd}+\mathrm{cd}a+\mathrm{d}a\mathrm{b}$$

$$\Leftrightarrow\sqrt{\mathrm{bc}}.(a\mathrm{b}+\mathrm{cd})\overset{?}{\geq}\mathrm{bc}(a+\mathrm{d})\Leftrightarrow a\mathrm{b}+\mathrm{cd}\underset{(2)}{\overset{?}{\geq}}\sqrt{\mathrm{bc}}.(a+\mathrm{d})\text{ and }\because\sqrt{\mathrm{bc}}\overset{\mathrm{AM-GM}}{\leq}\frac{\mathrm{b}+\mathrm{c}}{2}$$

$$\therefore\text{ in order to prove (2), it suffices to prove : }2(a\mathrm{b}+\mathrm{cd})\overset{?}{\geq}(\mathrm{b}+\mathrm{c})(a+\mathrm{d})$$

$$=a\mathrm{b}+a\mathrm{c}+\mathrm{bd}+\mathrm{cd}\Leftrightarrow a\mathrm{b}+\mathrm{cd}\overset{?}{\geq}a\mathrm{c}+\mathrm{bd}\to\text{true}\Rightarrow(2)\Rightarrow(*)\text{ is true}$$

$$\therefore\text{ combining both cases, }(*)\text{ is true }\forall\,a,\mathrm{b},\mathrm{c},\mathrm{d}>0\text{ and so,}$$

$$\frac{a\mathrm{bc}+\mathrm{bcd}+\mathrm{cd}a+\mathrm{d}a\mathrm{b}}{a+\mathrm{b}+\mathrm{c}+\mathrm{d}}\leq\frac{1}{4}.\sqrt[3]{(a+\mathrm{b})(a+\mathrm{c})(a+\mathrm{d})(\mathrm{b}+\mathrm{c})(\mathrm{b}+\mathrm{d})(\mathrm{c}+\mathrm{d})}$$

$$\forall\,a,\mathrm{b},\mathrm{c},\mathrm{d}>0\text{ and it's shown in the beginning that :}$$

$$\frac{1}{2}.\sqrt[6]{(a+\mathrm{b})(a+\mathrm{c})(a+\mathrm{d})(\mathrm{b}+\mathrm{c})(\mathrm{b}+\mathrm{d})(\mathrm{c}+\mathrm{d})}\leq\frac{a+\mathrm{b}+\mathrm{c}+\mathrm{d}}{4}$$

$$\Rightarrow\frac{1}{4}.\sqrt[3]{(a+\mathrm{b})(a+\mathrm{c})(a+\mathrm{d})(\mathrm{b}+\mathrm{c})(\mathrm{b}+\mathrm{d})(\mathrm{c}+\mathrm{d})}\leq\frac{(a+\mathrm{b}+\mathrm{c}+\mathrm{d})^2}{16}$$

$$\forall\,a,\mathrm{b},\mathrm{c},\mathrm{d}>0\text{ and the proof is complete, }''=''\text{ iff }a=\mathrm{b}=\mathrm{c}=\mathrm{d}\ (\mathrm{QED})$$