ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c}, \mathrm{d} > 0$, *then prove that* :

$$\frac{abc+\mathrm{bcd}+\mathrm{cd}a+\mathrm{d}ab}{a+\mathrm{b}+\mathrm{c}+\mathrm{d}} \le \frac{1}{4}\cdot\sqrt[3]{(a+\mathrm{b})(a+\mathrm{c})(a+\mathrm{d})(\mathrm{b}+\mathrm{c})(\mathrm{b}+\mathrm{d})(\mathrm{c}+\mathrm{d})}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$3(a+\mathrm{b}+\mathrm{c}+\mathrm{d}) =$$

$$= (a+\mathrm{b})+(a+\mathrm{c})+(a+\mathrm{d})+(\mathrm{b}+\mathrm{c})+(\mathrm{b}+\mathrm{d})+(\mathrm{c}+\mathrm{d}) \overset{\text{AM-GM}}{\ge}$$

$$\ge 6.\sqrt[6]{(a+\mathrm{b})(a+\mathrm{c})(a+\mathrm{d})(\mathrm{b}+\mathrm{c})(\mathrm{b}+\mathrm{d})(\mathrm{c}+\mathrm{d})}$$

$$\Rightarrow (a+\mathrm{b}+\mathrm{c}+\mathrm{d}).\frac{1}{4}.\sqrt[3]{(a+\mathrm{b})(a+\mathrm{c})(a+\mathrm{d})(\mathrm{b}+\mathrm{c})(\mathrm{b}+\mathrm{d})(\mathrm{c}+\mathrm{d})} \ge$$

$$\frac{1}{2}.\sqrt{(a+\mathrm{b})(a+\mathrm{c})(a+\mathrm{d})(\mathrm{b}+\mathrm{c})(\mathrm{b}+\mathrm{d})(\mathrm{c}+\mathrm{d})}$$

$$= \frac{1}{2}.\sqrt{\big((a+\mathrm{b})(\mathrm{b}+\mathrm{c})(\mathrm{c}+\mathrm{d})\big).\big((a+\mathrm{c})(a+\mathrm{d})(\mathrm{b}+\mathrm{d})\big)}$$

$$= \frac{\left(\sqrt{\begin{matrix}(abc+\mathrm{bcd}+\mathrm{cd}a+\mathrm{d}ab)\\ +\big(\mathrm{c}^2(a+\mathrm{b})+\mathrm{b}^2(\mathrm{c}+\mathrm{d})\big)\end{matrix}}\right).\left(\sqrt{\begin{matrix}(abc+\mathrm{bcd}+\mathrm{cd}a+\mathrm{d}ab)\\ +\big(a^2(\mathrm{b}+\mathrm{d})+\mathrm{d}^2(a+\mathrm{c})\big)\end{matrix}}\right)}{2}$$

$$\overset{\text{Reverse CBS}}{\ge} \frac{abc+\mathrm{bcd}+\mathrm{cd}a+\mathrm{d}ab+\sqrt{\big(\mathrm{c}^2(a+\mathrm{b})+\mathrm{b}^2(\mathrm{c}+\mathrm{d})\big).\big(a^2(\mathrm{b}+\mathrm{d})+\mathrm{d}^2(a+\mathrm{c})\big)}}{2}$$

$$\overset{?}{\ge} abc+\mathrm{bcd}+\mathrm{cd}a+\mathrm{d}ab$$

$$\Leftrightarrow \sqrt{\big(\mathrm{c}^2(a+\mathrm{b})+\mathrm{b}^2(\mathrm{c}+\mathrm{d})\big).\big(a^2(\mathrm{b}+\mathrm{d})+\mathrm{d}^2(a+\mathrm{c})\big)} \boxed{\underset{(*)}{\overset{?}{\ge}}} abc+\mathrm{bcd}+\mathrm{cd}a+\mathrm{d}ab$$

$\boxed{\text{Case 1}}$ $a\mathrm{c}+\mathrm{bd} \ge a\mathrm{b}+\mathrm{cd}$ *and then* : LHS of $(*) \overset{\text{Reverse CBS}}{\ge}$

$$\mathrm{c}a.\sqrt{(a+\mathrm{b})(\mathrm{b}+\mathrm{d})}+\mathrm{bd}.\sqrt{(\mathrm{c}+\mathrm{d})(a+\mathrm{c})}$$

$$= \mathrm{c}a.\sqrt{(\mathrm{b}+a)(\mathrm{b}+\mathrm{d})}+\mathrm{bd}.\sqrt{(\mathrm{c}+\mathrm{d})(\mathrm{c}+a)} \overset{\text{Reverse CBS}}{\ge}$$

$$\mathrm{c}a.\big(\mathrm{b}+\sqrt{a\mathrm{d}}\big)+\mathrm{bd}.\big(\mathrm{c}+\sqrt{a\mathrm{d}}\big) \overset{?}{\ge} abc+\mathrm{bcd}+\mathrm{cd}a+\mathrm{d}ab$$

$$\Leftrightarrow \sqrt{a\mathrm{d}}.(\mathrm{c}a+\mathrm{bd}) \overset{?}{\ge} a\mathrm{d}(\mathrm{b}+\mathrm{c}) \Leftrightarrow \mathrm{c}a+\mathrm{bd} \underset{(1)}{\overset{?}{\ge}} \sqrt{a\mathrm{d}}.(\mathrm{b}+\mathrm{c}) \text{ and } \because \sqrt{a\mathrm{d}} \overset{\text{AM-GM}}{\le} \frac{a+\mathrm{d}}{2}$$

$\therefore$ in order to prove (1), it suffices to prove : $2(a\mathrm{c}+\mathrm{bd}) \overset{?}{\ge} (a+\mathrm{d})(\mathrm{b}+\mathrm{c})$

$$= a\mathrm{b}+a\mathrm{c}+\mathrm{bd}+\mathrm{cd} \Leftrightarrow a\mathrm{c}+\mathrm{bd} \overset{?}{\ge} a\mathrm{b}+\mathrm{cd} \to \text{true} \Rightarrow (1) \Rightarrow (*) \text{ is true}$$

$\boxed{\text{Case 2}}$ $a\mathrm{b}+\mathrm{cd} \ge a\mathrm{c}+\mathrm{bd}$ *and then* : LHS of $(*) \overset{\text{Reverse CBS}}{\ge}$

$$\mathrm{cd}.\sqrt{(a+\mathrm{b})(a+\mathrm{c})}+a\mathrm{b}.\sqrt{(\mathrm{b}+\mathrm{d})(\mathrm{c}+\mathrm{d})}$$

$$= cd.\sqrt{(a+b)(a+c)} + ab.\sqrt{(d+b)(d+c)} \overset{\text{Reverse CBS}}{\geq}$$

$$cd.\left(a+\sqrt{bc}\right) + ab.\left(d+\sqrt{bc}\right) \overset{?}{\geq} abc + bcd + cda + dab$$

$$\Leftrightarrow \sqrt{bc}.(ab+cd) \overset{?}{\geq} bc(a+d) \Leftrightarrow ab + cd \underset{②}{\overset{?}{\geq}} \sqrt{bc}.(a+d) \text{ and } \because \sqrt{bc} \overset{\text{AM-GM}}{\leq} \frac{b+c}{2}$$

$$\therefore \text{ in order to prove ②, it suffices to prove : } 2(ab+cd) \overset{?}{\geq} (b+c)(a+d)$$

$$= ab + ac + bd + cd \Leftrightarrow ab + cd \overset{?}{\geq} ac + bd \to \text{true} \Rightarrow ② \Rightarrow (*) \text{ is true}$$

$$\therefore \text{ combining both cases, } (*) \text{ is true } \forall\, a,b,c,d > 0 \text{ and so,}$$

$$\frac{abc+bcd+cda+dab}{a+b+c+d} \leq \frac{1}{4}.\sqrt[3]{(a+b)(a+c)(a+d)(b+c)(b+d)(c+d)}$$

$$\forall\, a,b,c,d > 0, \text{"="} \text{ iff } a = b = c = d \text{ (QED)}$$