ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0, a^4 + b^4 + c^4 = 3$ then:

$$(a^3 + b^3)^c (b^3 + c^3)^a (c^3 + a^3)^b \le 8$$

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$$\frac{(\sum a^2)^2}{3} \overset{CBS}{\le} (a^4 + b^4 + c^4) = 3 \ or \sum a^2 \le 3 \ (1)$$

$$\left(\sum a^4\right)\left(\sum a^2\right) \overset{C-S}{\ge} \left(\sum a^3\right)^2 \ or \ 3 \times 3 \overset{(1)}{\ge} \left(\sum a^3\right)^2$$

$$\sum a^3 \le 3 \ (2)$$

$$\sum a^4 \overset{Chebyshev}{\ge} \frac{1}{3}\left(\sum a^3\right)\left(\sum a\right) \ (3)$$

$$\sum a \le \sqrt{3\left(\left(\sum a^2\right)\right)} \overset{(1)}{\le} 3 \ (4)$$

$$\sum a(b^3 + c^3) = \left(\sum a\right)\left(\sum a^3\right) - \left(\sum a^4\right) \overset{(3)}{\le} \frac{2}{3}\left(\sum a\right)\left(\sum a^3\right)$$

$$(a^3 + b^3)^c (b^3 + c^3)^a (c^3 + a^3)^b \overset{AM-GM}{\le} \left(\frac{\sum a(b^3 + c^3)}{a + b + c}\right)^{a+b+c} \le$$

$$\le \left(\frac{\frac{2}{3}(\sum a)(\sum a^3)}{a + b + c}\right)^{a+b+c} \overset{(2)\&(4)}{\le} 2^3 = 8$$

Equality holds for a=b=c=1.