ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0,\ a^2+b^2+c^2=3$ then:

$$(a^2+ab+b^2)^c(b^2+bc+c^2)^a(c^2+ca+a^2)^b\le 27$$

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Solution by Tapas Das-India

$$3=\ a^2+b^2+c^2=\frac{a^2}{1}+\frac{b^2}{1}+\frac{c^2}{1}\overset{Radon}{\ge}\frac{(a+b+c)^2}{3}\ or, a+b+c\le 3\ (1)$$

$$\sum c(a^2+ab+b^2)=\sum c(a^2+b^2)+3abc=\sum a\sum a^2-\sum a^3+3abc\overset{AM-GM}{\le}$$

$$\le\sum a\sum a^2-3abc+3abc=\sum a\sum a^2$$

$$(a^2+ab+b^2)^c(b^2+bc+c^2)^a(c^2+ca+a^2)^b\overset{AM-GM}{\le}$$

$$\le\left(\frac{\sum c(a^2+ab+b^2)}{a+b+c}\right)^{a+b+c}\le\left(\frac{\sum a\sum a^2}{a+b+c}\right)^{a+b+c}\overset{(1)}{\le}(3)^3=27$$

Equality holds for a=b=c=1.