ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > \frac{1}{3}$ and $\frac{1}{a^2(3b-1)} + \frac{1}{b^2(3a-1)} \geq \sqrt[3]{ab}$ then prove that :

$$1 + a + b \geq 3ab$$

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Let $x = a + b$ and $y = ab$ and when : $y \leq 1$, $a + b \overset{AM-GM}{\geq} 2\sqrt{y} = 2Y$ (say) $\overset{?}{\geq} 3ab - 1 = 3Y^2 - 1 \Leftrightarrow (3Y+1)(Y-1) \overset{?}{\leq} 0 \to$ true $\because Y = \sqrt{y} \leq 1$ and $a, b > \frac{1}{3}$ $\Rightarrow Y > \frac{1}{3} > 0 \therefore 1 + a + b \geq 3ab \;\forall\, y \in \left(\frac{1}{9}, 1\right]$ & we now focus on the case when : $y > 1$; and now, $\frac{1}{a^2(3b-1)} + \frac{1}{b^2(3a-1)} \geq \sqrt[3]{ab} \Rightarrow \frac{3xy - x^2 + 2y}{y^2(9y - 3x + 1)} \geq \sqrt[3]{y}$

$$\Rightarrow \frac{3xz^3 - x^2 + 2z^3}{9z^9 - 3xz^6 + z^6} \geq z\left(z = \sqrt[3]{y}\right) \Rightarrow x^2 - x(3z^7 + 3z^3) + 9z^{10} + z^7 - 2z^3 \overset{(1)}{\leq} 0$$

and discriminant of LHS of (1) $= \delta = (3z^7 + 3z^3)^2 - 4(9z^{10} + z^7 - 2z^3)$

$$= z^3(9z^{11} - 18z^7 - 4z^4 + 9z^3 + 8) = z^3\left(9z^3(z^4-1)^2 + 4(2 - z^4)\right) > 0$$

whenever : $z^4 \leq 2$ ($\because z > 0$) and when : $z^4 > 2$,

$$\delta = z^3\left((z^4 - 2)(9(z^7 - 1) + 5) + 9z^3\right) > 0 \text{ (since } z^4 > 2 > 1 \Rightarrow z > 1)$$

$$\therefore \delta > 0 \;\forall\, z > 0 \text{ (impling } \forall\, z > 1) \therefore (1) \Rightarrow x \geq \frac{3z^7 + 3z^3 - \sqrt{\delta}}{2} \overset{?}{>} 3ab - 1$$

$$= 3y - 1 = 3z^3 - 1 \Leftrightarrow 3z^7 - 3z^3 + 2 \overset{?}{>} \sqrt{\delta} \Leftrightarrow (3z^7 - 3z^3 + 2)^2 \overset{?}{>} \delta$$

$$= z^3(9z^{11} - 18z^7 - 4z^4 + 9z^3 + 8) \Leftrightarrow 4z^7 - 5z^3 + 1 \overset{?}{>} 0$$

$$\Leftrightarrow (z-1)\left(4z^6 + 4z^5 + 4z^4 + 4z^3 - z^2 - z - 1\right) \overset{?}{>} 0 \to \text{true} \because z = \sqrt[3]{y} > 1$$

$\therefore a + b > 3ab - 1 \;\forall\, y > 1$ and so, combining both cases, $1 + a + b \geq 3ab$ $\forall\, a, b > \frac{1}{3}$ and $\frac{1}{a^2(3b-1)} + \frac{1}{b^2(3a-1)} \geq \sqrt[3]{ab}$, " = " iff $a = b = 1$ (QED)