ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $a^2+b^2+c^2=3$ then prove that :

$$a^{12}+b^{12}+c^{12}+(abc)^2(a^3b^3+b^3c^3+c^3a^3)\ge 6$$

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$$\sum_{cyc}a^{12}+(abc)^2\left(\sum_{cyc}a^3b^3\right)\overset{AM-GM}{\ge}\sum_{cyc}a^{12}+3(abc)^4\overset{?}{\ge}6=$$

$$\frac{6}{729}\cdot\left(\sum_{cyc}a^2\right)^6\Leftrightarrow 243\sum_{cyc}x^6+729x^2y^2z^2\boxed{\underset{(*)}{\overset{?}{\ge}}}2\left(\sum_{cyc}x\right)^6\ (a^2=x,b^2=y,c^2=z)$$

Assigning $y+z=X, z+x=Y, x+y=Z\Rightarrow X+Y-Z=2z>0$,
$Y+Z-X=2x>0$ and $Z+X-Y=2y>0\Rightarrow X+Y>Z, Y+Z>X, Z+X>Y$
$\Rightarrow X,Y,Z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}x=\sum_{cyc}X=2s\Rightarrow\sum_{cyc}x=s\Rightarrow x=s-X, y=s-Y$,

$z=s-Z\therefore xyz=r^2s, \sum_{cyc}xy=4Rr+r^2, \sum_{cyc}x^3y^3=(4Rr+r^2)^3-12Rr^3s^2$,

$$\sum_{cyc}x^3=s^3-12Rrs\Rightarrow\sum_{cyc}x^6=(s^3-12Rrs)^2-2((4Rr+r^2)^3-12Rr^3s^2)$$

and so, $(*)\Leftrightarrow 243\left((s^3-12Rrs)^2-2((4Rr+r^2)^3-12Rr^3s^2)\right)+729r^4s^2-2s^6$
$\underset{(**)}{\overset{?}{\ge}}0$ and $\because 241(s^2-16Rr+5r^2)^3\overset{Gerretsen}{\ge}0\therefore$ in order to prove $(**)$, it

suffices to prove : LHS of $(**)\overset{?}{\ge}241(s^2-16Rr+5r^2)^3\Leftrightarrow(5736R-3615r)s^4-$
$r(16(9381R^2-7595Rr+1084r^2)+8Rr+2r^2)s^2+$
$r^2(128(7469R^3-7413R^2r+2214Rr^2-239r^3)+96R^2r-24Rr^2-19r^3)\boxed{\underset{(***)}{\overset{?}{\ge}}}0$

and $\because(5736R-3615r)(s^2-16Rr+5r^2)^2\overset{Gerretsen}{\ge}0\therefore$ in order to prove $(***)$,
it suffices to prove : LHS of $(***)\overset{?}{\ge}(5736R-3615r)(s^2-16Rr+5r^2)^2$

$$\Leftrightarrow(8364R^2-12882Rr+4701r^2)s^2\boxed{\underset{(****)}{\overset{?}{\ge}}}$$
$$r(32(4003R^3-6988R^2r+3425Rr^2-467r^3)+8R^2r+8Rr^2+3r^3)$$

and indeed, $(8364R^2-12882Rr+4701r^2)s^2\overset{Gerretsen}{\ge}$
$(8364R^2-12882Rr+4701r^2)(16Rr-5r^2)\overset{?}{\ge}$ RHS of $(****)$

$$\Leftrightarrow 2864t^3 - 12162t^2 + 15009t - 4282 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\big((t-2)(2864t-706)+729\big) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (****) \Rightarrow (***)$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore a^{12}+b^{12}+c^{12}+(abc)^2(a^3b^3+b^3c^3+c^3a^3) \geq 6$$

$$\forall a,b,c>0 \,\Big|\, a^2+b^2+c^2=3, ''='' \text{ iff } a=b=c=1 \text{ (QED)}$$