ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c,d>0,\ \frac{1}{a+3}+\frac{1}{b+3}+\frac{1}{c+3}+\frac{1}{d+3}=1$ then:

$$\sqrt{ab}+\sqrt{ac}+\sqrt{ad}+\sqrt{bc}+\sqrt{bd}+\sqrt{cd}\le 6$$

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Solution by Tapas Das-India

$$Let\ x_1=\sqrt{a}, x_2=\sqrt{b},\ x_3=\sqrt{c}, x_4=\sqrt{d}$$

$$\sqrt{ab}+\sqrt{ac}+\sqrt{ad}+\sqrt{bc}+\sqrt{bd}+\sqrt{cd}=x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4$$

$$Given\ \frac{1}{a+3}+\frac{1}{b+3}+\frac{1}{c+3}+\frac{1}{d+3}=1\ or\ \sum\frac{1}{a+3}=\sum\frac{1}{x_1^2+3}=1$$

$$\sum\frac{x_1^2}{x_1^2+3}=\sum\left(1-\frac{3}{x_1^2+3}\right)=4-3\sum\frac{1}{x_1^2+3}=4-3\times1=1$$

$$\left(\sum x_1\right)^2=\left(\sum\frac{x_1}{\sqrt{x_1^2+3}}.\sqrt{x_1^2+3}\right)^2\overset{Cauchy-Schwarz}{\le}\sum\frac{x_1^2}{x_1^2+3}.\sum(x_1^2+3)$$

$$or\ \left(\sum x_1\right)^2\le 1\times\left(3\times4+\sum x_1^2\right)\ or\ \left(\sum x_1\right)^2-\sum x_1^2\le 12$$

$$\sum x_1x_2=\frac{1}{2}\left(\left(\sum x_1\right)^2-\sum x_1^2\right)\le\frac{1}{2}\times12=6$$

$$\sqrt{ab}+\sqrt{ac}+\sqrt{ad}+\sqrt{bc}+\sqrt{bd}+\sqrt{cd}=$$
$$=x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4\le 6$$

Equality holds for a=b=c=d=1.