ROMANIAN MATHEMATICAL MAGAZINE

If $\sqrt{x^2+1}+\sqrt{y^2+1}-x-y=2\sqrt{2}-2$ then:

$$\sqrt{(x^2+1)(y^2+1)}+xy\ge 3$$

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$$Let\ a=\sqrt{x^2+1}-x, b=\sqrt{y^2+1}-y$$

$$\frac{1}{a}=\frac{1}{\sqrt{x^2+1}-x}=\sqrt{x^2+1}+x\ \ then\ a-\frac{1}{a}=-2x\ or\ x=\frac{1-a^2}{2a}$$

$$\sqrt{x^2+1}=x+a=\frac{1-a^2}{2a}+a=\frac{1+a^2}{2a}$$

$$Similary:\ y=\frac{1-b^2}{2b},\sqrt{y^2+1}=\frac{1+b^2}{2b}$$

$$\sqrt{(x^2+1)(y^2+1)}+xy=\frac{1+a^2}{2a}\cdot\frac{1+b^2}{2b}+\frac{1-a^2}{2a}\frac{1-b^2}{2b}=\frac{1+a^2b^2}{2ab}(1)$$

$$\sqrt{x^2+1}+\sqrt{y^2+1}-x-y=2\sqrt{2}-2\ \ or\ a+b=2\left(\sqrt{2}-1\right)$$

$$ab\overset{AM-GM}{\le}\left(\frac{a+b}{2}\right)^2=\left(\frac{2\left(\sqrt{2}-1\right)}{2}\right)^2=3-2\sqrt{2}\ \ or\ \ 2\sqrt{2}\le 3-ab$$

$$or,\left(2\sqrt{2}\right)^2\le(3-ab)^2\ or\ \ 8\le 9-6ab+a^2b^2\ or\ \ a^2b^2-6ab+1\ge 0\ (2)$$

$$We\ need\ to\ show:\sqrt{(x^2+1)(y^2+1)}+xy\ge 3$$

$$or\ \frac{1+a^2b^2}{2ab}\ge 3\ \left(using\ (1)\right)\ or\ a^2b^2-6ab+1\ge 0\ \ this\ is\ true\ by\ (2)$$

$$Equality\ holds\ for\ a=b=\sqrt{2}-1\ or\ x=y=1.$$