ROMANIAN MATHEMATICAL MAGAZINE

If $\sqrt{x^2+1}+\sqrt{y^2+1}+x+y=2\sqrt{2}+2$ then:

$$x\sqrt{y^2+1}+y\sqrt{x^2+1}\le 2\sqrt{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ u=\sqrt{x^2+1}+x, v=\sqrt{y^2+1}+y\ then\ u+v=2\sqrt{2}+2=2\left(\sqrt{2}+1\right)\ (1)$$

$$u=\sqrt{x^2+1}+x, \frac{1}{u}=\sqrt{x^2+1}-x\ then\ u-\frac{1}{u}=2x\ or\ x=\frac{u^2-1}{2u}\ and$$

$$\sqrt{x^2+1}=u-x=\frac{u^2+1}{2u}$$

$$Similarly:\ y=\frac{v^2-1}{2v}, \sqrt{y^2+1}=\frac{v^2+1}{2v}$$

$$x\sqrt{y^2+1}+y\sqrt{x^2+1}=\frac{u^2-1}{2u}.\frac{v^2+1}{2v}+\frac{v^2-1}{2v}.\frac{u^2+1}{2u}=\frac{u^2v^2-1}{2uv}\ (2)$$

$$uv\overset{AM-GM}{\le}\left(\frac{u+v}{2}\right)^2\overset{(1)}{=}\left(\frac{2\left(\sqrt{2}+1\right)}{2}\right)^2=3+2\sqrt{2}\ (3)$$

$$We\ need\ to\ show:$$

$$x\sqrt{y^2+1}+y\sqrt{x^2+1}\le 2\sqrt{2}$$

$$or\ \frac{u^2v^2-1}{2uv}\le 2\sqrt{2}\ \ or\ u^2v^2-4\sqrt{2}uv-1\le 0$$

$$or\ \left(uv-\left(2\sqrt{2}+3\right)\right)\left(uv-\left(2\sqrt{2}-3\right)\right)\le 0$$

$$which\ implies\ 2\sqrt{2}-3\le\ uv\le\ 2\sqrt{2}+3\ this\ is\ true\ by\ (3)$$

Equality holds for x=y=1.