ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ then prove that :

$$\left(\sum_{cyc} a^8\right)\left(\sum_{cyc} \frac{1}{a^8}\right) \geq \left(\sum_{cyc} a\right)\left(\sum_{cyc} \frac{1}{a}\right)$$

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$$\left(\sum_{cyc} a^8\right)\left(\sum_{cyc} \frac{1}{a^8}\right) \overset{?}{\geq} \left(\sum_{cyc} a\right)\left(\sum_{cyc} \frac{1}{a}\right)$$

$$\Leftrightarrow 3 + \sum_{cyc} \frac{a^8}{b^8} + \sum_{cyc} \frac{b^8}{a^8} \overset{?}{\geq} 3 + \sum_{cyc} \frac{a}{b} + \sum_{cyc} \frac{b}{a} \Leftrightarrow \sum_{cyc} \frac{a^8}{b^8} + \sum_{cyc} \frac{b^8}{a^8} \overset{?}{\geq} \sum_{cyc} \frac{a}{b} + \sum_{cyc} \frac{b}{a} \rightarrow (1)$$

$$\text{Now, } \sum_{cyc} \frac{a^8}{b^8} + \sum_{cyc} \frac{b^8}{a^8} \overset{\text{Holder}}{\geq} \frac{1}{3^7}.\left(\sum_{cyc} \frac{a}{b}\right)^8 + \frac{1}{3^7}.\left(\sum_{cyc} \frac{b}{a}\right)^8 \overset{\text{AM-GM}}{\geq}$$

$$\frac{1}{3^7}.\left(\sum_{cyc} \frac{a}{b}\right)\left(3.\sqrt[3]{\frac{a}{b}.\frac{b}{c}.\frac{c}{a}}\right)^7 + \frac{1}{3^7}.\left(\sum_{cyc} \frac{b}{a}\right)\left(3.\sqrt[3]{\frac{b}{a}.\frac{a}{c}.\frac{c}{b}}\right)^7$$

$$= \frac{1}{3^7}.\left(\sum_{cyc} \frac{a}{b}\right).3^7 + \frac{1}{3^7}.\left(\sum_{cyc} \frac{b}{a}\right).3^7 = \sum_{cyc} \frac{a}{b} + \sum_{cyc} \frac{b}{a} \Rightarrow (1) \text{ is true}$$

$$\therefore \left(\sum_{cyc} a^8\right)\left(\sum_{cyc} \frac{1}{a^8}\right) \geq \left(\sum_{cyc} a\right)\left(\sum_{cyc} \frac{1}{a}\right) \forall a, b, c > 0, \text{''}=\text{''} \text{ iff } a = b = c \text{ (QED)}$$