ROMANIAN MATHEMATICAL MAGAZINE

If $a_k > 0$ then prove that :

$$\sum_{cyc} \frac{8a_1^3}{(a_1+a_2)(\sqrt{5}a_1+a_2)} \geq (\sqrt{5}-1)\sum_{k=1}^{n} a_k$$

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$$\frac{8a_1^3}{(a_1+a_2)(\sqrt{5}a_1+a_2)} - (\sqrt{5}a_1 - a_2) =$$

$$= \frac{8a_1^3 - (a_1+a_2)(\sqrt{5}a_1+a_2)(\sqrt{5}a_1-a_2)}{(a_1+a_2)(\sqrt{5}a_1+a_2)} = \frac{8a_1^3 - (a_1+a_2)(5a_1^2-a_2^2)}{(a_1+a_2)(\sqrt{5}a_1+a_2)}$$

$$= \frac{3a_1^3 - 5a_1^2a_2 + a_1a_2^2 + a_2^3}{(a_1+a_2)(\sqrt{5}a_1+a_2)} = \frac{(a_1-a_2)^2(3a_1+a_2)}{(a_1+a_2)(\sqrt{5}a_1+a_2)} \geq 0$$

$$\therefore \frac{8a_1^3}{(a_1+a_2)(\sqrt{5}a_1+a_2)} \geq \sqrt{5}a_1 - a_2 \text{ and analogs}$$

$$\Rightarrow \sum_{cyc} \frac{8a_1^3}{(a_1+a_2)(\sqrt{5}a_1+a_2)} \geq \sum_{cyc}(\sqrt{5}a_1 - a_2) = \sqrt{5}\sum_{k=1}^{n} a_k - \sum_{k=1}^{n} a_k$$

$$= (\sqrt{5}-1)\sum_{k=1}^{n} a_k \therefore \sum_{cyc} \frac{8a_1^3}{(a_1+a_2)(\sqrt{5}a_1+a_2)} \geq (\sqrt{5}-1)\sum_{k=1}^{n} a_k \ \forall a_k > 0,$$

$$k = \overline{1,n}, '' ='' \text{ iff } a_1 = a_2 = \cdots a_n \text{ (QED)}$$