ROMANIAN MATHEMATICAL MAGAZINE

Let $n \geq 2$ and $x_1, x_2, \dots, x_n$ be positive real numbers. Prove that :

$$\sqrt{\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^2} < \sum_{i=1}^{n}\frac{1}{x_i} - \frac{1}{\sum_{i=1}^{n} x_i}$$

Proposed by Mehmet Şahin-Turkiye

Solution by Soumava Chakraborty-Kolkata-India

$$\sqrt{\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^2} + \frac{1}{\sum_{i=1}^{n} x_i} < \sqrt{\left(\sum_{i=1}^{n}\frac{1}{x_i}\right)^2} + \frac{1}{\sum_{i=1}^{n} x_i} = \frac{2}{n^2}\cdot\frac{n^2}{\sum_{i=1}^{n} x_i} \leq$$

$$\overset{\text{Reverse Bergstrom}}{\leq} \frac{2}{n^2}\cdot\sum_{i=1}^{n}\frac{1}{x_i} \overset{n\geq 2}{\leq} \frac{2}{4}\cdot\sum_{i=1}^{n}\frac{1}{x_i} < \sum_{i=1}^{n}\frac{1}{x_i}$$

$$\therefore \sqrt{\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^2} < \sum_{i=1}^{n}\frac{1}{x_i} - \frac{1}{\sum_{i=1}^{n} x_i} \ \forall\, n \geq 2 \text{ and } \forall\, x_1, x_2, \dots, x_n > 0 \ (\text{QED})$$