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If $a,b>0$ and $n\in\mathbb{N}, n\ge 2$ then :

$$\sqrt[n]{\frac{a}{b}}+\sqrt[n]{\frac{b}{a}}\le\sqrt[n]{2^{n-2}(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)}$$

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$$\text{Let } \frac{a}{b}=t \text{ and then : } \sqrt[n]{\frac{a}{b}}+\sqrt[n]{\frac{b}{a}}\overset{?}{\le}\sqrt[n]{2^{n-2}(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)}$$

$$\Leftrightarrow \sqrt[n]{t}+\frac{1}{\sqrt[n]{t}}\overset{?}{\le}\sqrt[n]{2^{n-2}.\left(2+t+\frac{1}{t}\right)}\Leftrightarrow 1+t^{\frac{2}{n}}\overset{?}{\le}\sqrt[n]{2^{n-2}.(2t+t^2+1)}$$

$$=2^{\frac{n-2}{n}}.(t+1)^{\frac{2}{n}}\Leftrightarrow\left(\frac{1}{t+1}\right)^{\frac{2}{n}}+\left(\frac{t}{t+1}\right)^{\frac{2}{n}}\overset{?}{\le}2^{\frac{n-2}{n}}\Leftrightarrow\left(\frac{1}{t+1}\right)^{\frac{2}{n}}+\left(1-\frac{1}{t+1}\right)^{\frac{2}{n}}\overset{?}{\le}2^{\frac{n-2}{n}}$$

$$\Leftrightarrow u^{\frac{2}{n}}+(1-u)^{\frac{2}{n}}\boxed{\underset{(*)}{\overset{?}{\le}}}2^{\frac{n-2}{n}}\left(u=\frac{1}{t+1};0<u<1\because t>0\right)$$

$$\text{Let } F(u)=u^{\frac{2}{n}}+(1-u)^{\frac{2}{n}}-2^{\frac{n-2}{n}}\ \forall\, u\in(0,1) \text{ and } \forall\, n\in(\mathbb{N}^*-\{1\}); n\to\text{fixed}$$

$$\text{and then : } F'(u)=\frac{2}{n}\left(u^{\frac{2}{n}-1}-(1-u)^{\frac{2}{n}-1}\right)$$

$$\Rightarrow F'(u)\overset{(\bullet)}{=}\frac{2}{n}.(1-u)^{\frac{2}{n}-1}.\left(\left(\frac{u}{1-u}\right)^{\frac{2}{n}-1}-1\right)$$

$$\boxed{\text{Case 1}}\ 0<u\le\frac{1}{2} \text{ and then : } 2u\le 1\Rightarrow u\le 1-u\Rightarrow\frac{u}{1-u}\le 1$$

$$\Rightarrow\left(\frac{2}{n}-1\right).\ln\frac{u}{1-u}\ge 0\ \left(\because\frac{2}{n}-1\le 0\right)\Rightarrow\ln\left(\left(\frac{u}{1-u}\right)^{\frac{2}{n}-1}\right)\ge 0$$

$$\Rightarrow\frac{2}{n}.(1-u)^{\frac{2}{n}-1}.\left(\left(\frac{u}{1-u}\right)^{\frac{2}{n}-1}-1\right)\ge 0\ \left(\because 0<u\le\frac{1}{2}<1\Rightarrow 1-u>0\right)\overset{\text{via }(\bullet)}{\Rightarrow}$$

$$F'(u)\ge 0\Rightarrow F(u)\text{ is }\uparrow\text{ on }\left(0,\frac{1}{2}\right]\Rightarrow F(u)\le F\left(\frac{1}{2}\right)=2\left(\frac{1}{2}\right)^{\frac{2}{n}}-2^{\frac{n-2}{n}}=0\ \forall\, u\in\left(0,\frac{1}{2}\right]$$

$$\boxed{\text{Case 2}}\ \frac{1}{2}\le u<1 \text{ and then : } 2u\ge 1\Rightarrow u\ge 1-u\Rightarrow\frac{u}{1-u}\ge 1$$

$$\Rightarrow\left(\frac{2}{n}-1\right).\ln\frac{u}{1-u}\le 0\ \left(\because\frac{2}{n}-1\le 0\right)\Rightarrow\ln\left(\left(\frac{u}{1-u}\right)^{\frac{2}{n}-1}\right)\le 0$$

$$\Rightarrow\frac{2}{n}.(1-u)^{\frac{2}{n}-1}.\left(\left(\frac{u}{1-u}\right)^{\frac{2}{n}-1}-1\right)\le 0\ \left(\because 0<u\le\frac{1}{2}<1\Rightarrow 1-u>0\right)\overset{\text{via }(\bullet)}{\Rightarrow}$$

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$$\mathbf{F}'(\mathbf{u}) \leq \mathbf{0} \Rightarrow \mathbf{F}(\mathbf{u}) \text{ is } \downarrow \text{ on } \left[\frac{1}{2}, 1\right) \Rightarrow \mathbf{F}(\mathbf{u}) \leq \mathbf{F}\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^{\frac{2}{n}} - 2^{\frac{n-2}{n}} = 0 \ \forall \ \mathbf{u} \in \left[\frac{1}{2}, 1\right)$$

$$\therefore \text{ combining cases 1 and 2}, \mathbf{F}(\mathbf{u}) \leq 0 \ \forall \ \mathbf{u} \in (0,1) \text{ and } \forall \ \mathbf{n} \in (\mathbb{N}^* - \{1\}); \mathbf{n} \to \text{fixed}$$

$$\Rightarrow (*) \text{ is true } \forall \ \mathbf{u} \in (0,1) \text{ and } \forall \ \mathbf{n} \in (\mathbb{N}^* - \{1\})$$

$$\therefore \sqrt[n]{\frac{a}{b}} + \sqrt[n]{\frac{b}{a}} \leq \sqrt[n]{2^{n-2}(a+b)\left(\frac{1}{a} + \frac{1}{b}\right)} \ \forall \ a, b > 0 \ \land \ n \in \mathbb{N} \,\middle|\, n \geq 2,$$

$$'' ='' \text{ iff } a = b \text{ (QED)}$$