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If $a,\mathrm{b},\mathrm{c},\mathrm{d},\mathrm{e}>0$ *and* $abcde=1$ then :

$$\sum_{cyc} a^5\mathrm{b} \geq \sum_{cyc} a$$

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$$a^3+\mathrm{b}^3+\mathrm{c}^3 \overset{\text{AM-GM}}{\geq} 3a\mathrm{bc},\ \mathrm{b}^3+\mathrm{c}^3+\mathrm{d}^3 \overset{\text{AM-GM}}{\geq} 3\mathrm{bcd},$$

$$\mathrm{c}^3+\mathrm{d}^3+\mathrm{e}^3 \overset{\text{AM-GM}}{\geq} 3\mathrm{cde},\ \mathrm{d}^3+\mathrm{e}^3+a^3 \overset{\text{AM-GM}}{\geq} 3\mathrm{de}a \text{ and } \mathrm{e}^3+a^3+\mathrm{b}^3 \overset{\text{AM-GM}}{\geq} 3\mathrm{e}a\mathrm{b}$$

and via summation, we get :

$$3(a^3+\mathrm{b}^3+\mathrm{c}^3+\mathrm{d}^3+\mathrm{e}^3) \geq 3(a\mathrm{bc}+\mathrm{bcd}+\mathrm{cde}+\mathrm{de}a+\mathrm{e}a\mathrm{b})$$

$$\Rightarrow \sum_{cyc} a^3 \boxed{\overset{①}{\geq}} \sum_{cyc} a\mathrm{bc} \ \&\ \textit{now},\ (a+\mathrm{b}+\mathrm{c}+\mathrm{d}+\mathrm{e})(a^2+\mathrm{b}^2+\mathrm{c}^2+\mathrm{d}^2+\mathrm{e}^2)$$

$$= \sum_{cyc} a^3 + a\left((\mathrm{b}^2+\mathrm{c}^2)+(\mathrm{d}^2+\mathrm{e}^2)\right)+\mathrm{b}\left((\mathrm{c}^2+\mathrm{d}^2)+(\mathrm{e}^2+a^2)\right)+$$

$$\mathrm{c}\left((\mathrm{d}^2+\mathrm{e}^2)+(a^2+\mathrm{b}^2)\right)+\mathrm{d}\left((\mathrm{e}^2+a^2)+(\mathrm{b}^2+\mathrm{c}^2)\right)+\mathrm{e}\left((a^2+\mathrm{b}^2)+(\mathrm{c}^2+\mathrm{d}^2)\right)$$

$$\overset{\text{AM-GM}}{\geq} \sum_{cyc} a^3 + a(2\mathrm{bc}+2\mathrm{de})+\mathrm{b}(2\mathrm{cd}+2\mathrm{e}a)+\mathrm{c}(2\mathrm{de}+2a\mathrm{b})+\mathrm{d}(2\mathrm{e}a+2\mathrm{bc})$$

$$\overset{\text{via } ①}{\geq} \sum_{cyc} a\mathrm{bc} + 4\sum_{cyc} a\mathrm{bc} \Rightarrow \sum_{cyc} a\mathrm{bc} \boxed{\overset{②}{\leq}} \frac{1}{5}\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right) \text{ and since } a\mathrm{bcde}=1,$$

$$\text{we have : we have : } \sum_{cyc} a^5\mathrm{b} = \sum_{cyc} \frac{a^4}{\mathrm{cde}} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc} a^2\right)^2}{\sum_{cyc} a\mathrm{bc}} \overset{\text{via } ②}{\geq} \frac{5\left(\sum_{cyc} a^2\right)^2}{\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right)}$$

$$= \frac{5\left(\sum_{cyc} a^2\right)}{\sum_{cyc} a} \overset{\text{Chebyshev}}{\geq} \frac{5.\frac{1}{5}\left(\sum_{cyc} a\right)^2}{\sum_{cyc} a} = \sum_{cyc} a \text{ and so, } \sum_{cyc} a^5\mathrm{b} \geq \sum_{cyc} a$$

$$\forall\, a,\mathrm{b},\mathrm{c},\mathrm{d},\mathrm{e}>0 \,\big|\, a\mathrm{bcde}=1, ''='' \text{ iff } a=\mathrm{b}=\mathrm{c}=\mathrm{d}=\mathrm{e}=1 \text{ (QED)}$$