ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, x + y + z = 3$ *and* $\lambda < \frac{9}{2}$ then :

$$\frac{\sum_{cyc} xy}{\sum_{cyc} x^3} + \frac{\lambda}{3} - 1 \leq \frac{\lambda}{\sum_{cyc} x^2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\sum_{cyc} xy}{\sum_{cyc} x^3} + \frac{\lambda}{3} - 1 \overset{?}{\leq} \frac{\lambda}{\sum_{cyc} x^2} \Leftrightarrow \lambda\left(\frac{1}{3} - \frac{1}{\sum_{cyc} x^2}\right) \overset{?}{\leq} 1 - \frac{\sum_{cyc} xy}{\sum_{cyc} x^3}$$

$$\Leftrightarrow \lambda\left(\frac{1}{3} - \frac{\left(\sum_{cyc} x\right)^2}{9\sum_{cyc} x^2}\right) \overset{?}{\leq} 1 - \frac{\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x\right)}{3\sum_{cyc} x^3} \; (\because x + y + z = 3)$$

$$\Leftrightarrow \lambda . \frac{3\sum_{cyc} x^2 - \left(\sum_{cyc} x\right)^2}{9\sum_{cyc} x^2} \overset{?}{\leq} \frac{3\sum_{cyc} x^3 - \left(\sum_{cyc} xy\right)\left(\sum_{cyc} x\right)}{3\sum_{cyc} x^3} \; \textit{and}$$

$$\because \lambda < \frac{9}{2} \wedge 3\sum_{cyc} x^2 - \left(\sum_{cyc} x\right)^2 \geq 0 \therefore \text{it suffices to prove :}$$

$$\frac{9}{2} . \frac{3\sum_{cyc} x^2 - \left(\sum_{cyc} x\right)^2}{9\sum_{cyc} x^2} \overset{?}{\leq} \frac{3\sum_{cyc} x^3 - \left(\sum_{cyc} xy\right)\left(\sum_{cyc} x\right)}{3\sum_{cyc} x^3}$$

$$\Leftrightarrow 2\sum_{cyc} x^4 y + 2\sum_{cyc} xy^4 \underset{(*)}{\overset{?}{\geq}} \sum_{cyc} x^3 y^2 + \sum_{cyc} x^2 y^3 + 2xyz\sum_{cyc} xy$$

$$\text{Now, } 2\sum_{cyc} x^4 y + 2\sum_{cyc} xy^4 = \sum_{cyc} xy(x^3 + y^3) + \sum_{cyc} z(x^4 + y^4)$$

$$\overset{AM-GM}{\geq} \sum_{cyc} x^2 y^2 (x + y) + \sum_{cyc} 2z(x^2 y^2) = \sum_{cyc} x^3 y^2 + \sum_{cyc} x^2 y^3 + 2xyz\sum_{cyc} xy$$

$$\Rightarrow (*) \text{ is true} \therefore \frac{\sum_{cyc} xy}{\sum_{cyc} x^3} + \frac{\lambda}{3} - 1 \leq \frac{\lambda}{\sum_{cyc} x^2} \forall x, y, z > 0 \,\middle|\, x + y + z = 3 \; \textit{and} \; \lambda < \frac{9}{2},$$

$$'' ='' \text{ iff } x = y = z = 1 \text{ (QED)}$$