ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z} \le 3m^2$ and $k, n \in \mathbb{N}^*, m > 0$ then :

$$\sum_{cyc} \frac{1}{\sqrt{(\lambda^2+1)x^2 + 2(\lambda n - 1)xy + (n^2+1)y^2}} \le \frac{3m}{\lambda + n}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{1}{\sqrt{(\lambda^2+1)x^2 + 2(\lambda n - 1)xy + (n^2+1)y^2}} =$$

$$= \sum_{cyc} \frac{1}{\sqrt{(x-y)^2 + (\lambda x + ny)^2}} \le \sum_{cyc} \frac{1}{\lambda x + ny} \overset{\text{Weighted AM}-\text{Weighted HM}}{\le} \sum_{cyc} \frac{\frac{\lambda}{x} + \frac{n}{y}}{(\lambda+n)^2}$$

$$= \frac{1}{(\lambda+n)^2} \cdot \left(\lambda \sum_{cyc} \frac{1}{x} + n \sum_{cyc} \frac{1}{x}\right) \overset{\text{CBS}}{\le} \frac{\lambda+n}{(\lambda+n)^2} \cdot \sqrt{3 \sum_{cyc} \frac{1}{x^2}} \le \frac{1}{\lambda+n} \cdot \sqrt{3.3m^2} = \frac{3m}{\lambda+n}$$

$$(\because m > 0) \therefore \sum_{cyc} \frac{1}{\sqrt{(\lambda^2+1)x^2 + 2(\lambda n - 1)xy + (n^2+1)y^2}} \le \frac{3m}{\lambda+n}$$

$$\forall\, x, y, z > 0 \,\Big|\, \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z} \le 3m^2 \wedge k, n \in \mathbb{N}^*, m > 0, '' ='' \text{ iff } x = y = z \text{ (QED)}$$