ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z>0, x+y+z=3$ and $\lambda\geq 3$ then :

$$\sum_{cyc} x^3y+3(\lambda-1)\geq\lambda\sum_{cyc} xy$$

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Assigning $y+z=a, z+x=b, x+y=c\Rightarrow a+b-c=2z>0, b+c-a=2x>0$ and $c+a-b=2y>0\Rightarrow a+b>c, b+c>a, c+a>b\Rightarrow a,b,c$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) $\Rightarrow \sum_{cyc} x=s, \sum_{cyc} xy=4Rr+r^2$ and $\sum_{cyc} x^3y+3(\lambda-1)-\lambda\sum_{cyc} xy$

$$\overset{x+y+z=3}{=}\sum_{cyc}\frac{x^3y^3}{y^2}+\frac{3(\lambda-1)}{81}.\left(\sum_{cyc} x\right)^4-\frac{\lambda}{9}\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x\right)^2$$

$$\overset{Radon}{\geq}\frac{\left(\sum_{cyc} xy\right)^3}{\left(\sum_{cyc} x\right)^2}+\frac{\lambda}{27}\left(\sum_{cyc} x\right)^2.\left(\left(\sum_{cyc} x\right)^2-3\sum_{cyc} xy\right)-\frac{1}{27}.\left(\sum_{cyc} x\right)^4$$

$$\overset{\lambda\geq 3}{\geq}\frac{\left(\sum_{cyc} xy\right)^3}{\left(\sum_{cyc} x\right)^2}+\frac{1}{9}\left(\sum_{cyc} x\right)^2.\left(\left(\sum_{cyc} x\right)^2-3\sum_{cyc} xy\right)-\frac{1}{27}.\left(\sum_{cyc} x\right)^4$$

$$\left(\because\left(\sum_{cyc} x\right)^2\geq 3\sum_{cyc} xy\right)\overset{?}{\geq}0\Leftrightarrow$$

$$27\left(\sum_{cyc} xy\right)^3+3\left(\sum_{cyc} x\right)^4.\left(\left(\sum_{cyc} x\right)^2-3\sum_{cyc} xy\right)-\left(\sum_{cyc} x\right)^6\overset{?}{\geq}0\overset{\text{via transformation}}{\Leftrightarrow}$$

$$2s^6-9(4Rr+r^2)s^4+27(4Rr+r^2)^3\ \boxed{\underset{(*)}{\overset{?}{\geq}}}\ 0 \text{ and } \because 2(s^2-16Rr+5r^2)^3\overset{Gerretsen}{\geq}0$$

$\therefore$ in order to prove $(*)$, it suffices to prove : LHS of $(*)\overset{?}{\geq}2(s^2-16Rr+5r^2)^3$

$$\Leftrightarrow(60R-39r)s^4-r(1536R^2-960Rr+150r^2)s^2+r^2(9920R^3-6384R^2r+2724Rr^2-223r^3)\ \boxed{\underset{(**)}{\overset{?}{\geq}}}\ 0 \text{ and}$$

$\because (60R-39r)(s^2-16Rr+5r^2)^2\overset{Gerretsen}{\geq}0\therefore$ to prove $(**)$, it suffices to prove : LHS of $(**)\overset{?}{\geq}(60R-39r)(s^2-16Rr+5r^2)^2$

$$\Leftrightarrow(48R^2-111Rr+30r^2)s^2\ \boxed{\underset{(***)}{\overset{?}{\geq}}}\ r(680R^3-1650R^2r+627Rr^2-94r^3) \text{ and}$$

$$\mathbf{finally}, (48R^2 - 111Rr + 30r^2)s^2 \overset{\text{Gerretsen}}{\geq} (48R^2 - 111Rr + 30r^2)(16Rr - 5r^2)$$

$$\overset{?}{\geq} \text{RHS of } (***) \Leftrightarrow 44t^3 - 183t^2 + 204t - 28 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)^2(44t-7) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (***) \Rightarrow (**) \Rightarrow (**) \text{ is true}$$

$$\therefore \sum_{\text{cyc}} x^3y + 3(\lambda - 1) \geq \lambda \sum_{\text{cyc}} xy \;\; \forall\, x, y, z > 0 \,\middle|\, x + y + z = 3 \text{ and } \lambda \geq 3,$$

$$'' ='' \text{ iff } x = y = z = 1 \text{ (QED)}$$