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**If $a, \mathrm{b}, \mathrm{c} > 0, a + \mathrm{b} + \mathrm{c} = 3$ and $\lambda \geq 5$, then :**

$$\sum_{cyc} a^2\mathrm{b}^2 + \lambda \sum_{cyc} a\mathrm{b} \leq 3(\lambda + 1)$$

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$$\sum_{cyc} a^2\mathrm{b}^2 + \lambda \sum_{cyc} a\mathrm{b} \leq 3(\lambda + 1) \Leftrightarrow \sum_{cyc} a^2\mathrm{b}^2 + \frac{\lambda}{9}\left(\sum_{cyc} a\mathrm{b}\right)\left(\sum_{cyc} a\right)^2 \leq$$

$$\frac{3(\lambda + 1)}{81}\left(\sum_{cyc} a\right)^4 \left(\because \sum_{cyc} a = 3\right)$$

$$\Leftrightarrow \lambda\left(\sum_{cyc} a\right)^2\left(\left(\sum_{cyc} a\right)^2 - 3\sum_{cyc} a\mathrm{b}\right) + \left(\sum_{cyc} a\right)^4 - 27\sum_{cyc} a^2\mathrm{b}^2 \geq 0 \text{ and}$$

$$\because \lambda \geq 5 \wedge \left(\sum_{cyc} a\right)^2 - 3\sum_{cyc} a\mathrm{b} \geq 0 \therefore \text{ it suffices to prove :}$$

$$5\left(\sum_{cyc} a\right)^2\left(\left(\sum_{cyc} a\right)^2 - 3\sum_{cyc} a\mathrm{b}\right) + \left(\sum_{cyc} a\right)^4 - 27\sum_{cyc} a^2\mathrm{b}^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow 2\sum_{cyc} a^4 + 3\sum_{cyc} a^3\mathrm{b} + 3\sum_{cyc} a\mathrm{b}^3 \overset{?}{\geq} 7\sum_{cyc} a^2\mathrm{b}^2 + a\mathrm{bc}\sum_{cyc} a \rightarrow \text{true}$$

$$\because 2\sum_{cyc} a^4 + 3\sum_{cyc} a^3\mathrm{b} + 3\sum_{cyc} a\mathrm{b}^3 \overset{AM-GM}{\geq} 2\sum_{cyc} a^2\mathrm{b}^2 + 6\sum_{cyc} a^2\mathrm{b}^2$$

$$\geq \sum_{cyc} a^2\mathrm{b}^2 + a\mathrm{bc}\sum_{cyc} a + 6\sum_{cyc} a^2\mathrm{b}^2 = 7\sum_{cyc} a^2\mathrm{b}^2 + a\mathrm{bc}\sum_{cyc} a \text{ and so,}$$

$$\sum_{cyc} a^2\mathrm{b}^2 + \lambda\sum_{cyc} a\mathrm{b} \leq 3(\lambda + 1)\ \forall\, a, \mathrm{b}, \mathrm{c} > 0 \,\middle|\, a + \mathrm{b} + \mathrm{c} = 3 \text{ and } \lambda \geq 5,$$

$$'' ='' \text{ iff } a = \mathrm{b} = \mathrm{c} = 1\ (\text{QED})$$