ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ then:

$$\sum_{cyc}(b+c)(a^2+b^2+ab) \geq \frac{9}{4}\prod_{cyc}(a+b)$$

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$Lemma$:

$$\forall\, x, y > 0,\ x^2+xy+y^2 \geq \frac{3}{4}(x+y)^2$$

$Proof$:

$$x^2+y^2+xy = (x+y)^2 - xy \overset{AM-GM}{\geq} (x+y)^2 - \frac{(x+y)^2}{4} = \frac{3}{4}(x+y)^2$$

$$\sum (b+c)(a^2+b^2+ab) \overset{Lemma}{\geq} \sum (b+c)\frac{3}{4}(a+b)^2 =$$

$$= \frac{3}{4}\sum (a+b)^2(b+c) \overset{AM-GM}{\geq} \frac{9}{4}\sqrt[3]{(a+b)^3(b+c)^3(c+a)^3} =$$

$$= \frac{9}{4}(a+b)(b+c)(c+a)$$

Equality holds for a=b=c.