ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0, a+b+c=3$ then:

$$a^2+b^2+c^2\ge abc\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)$$

Proposed by Gheorghe Crăciun-Romania

Solution by Tapas Das-India

Lemma:

$$If\ a,b,c>0\ \&\ a+b+c=3\ then\ \sqrt{a}+\sqrt{b}+\sqrt{c}\ge ab+bc+ca$$

Proof:

$$ab+bc+ca=\frac{1}{2}\left((a+b+c)^2-(a^2+b^2+c^2)\right)=$$

$$=\frac{1}{2}\left(3^2-(a^2+b^2+c^2)\right)=\frac{1}{2}\left(9-(a^2+b^2+c^2)\right)$$

$$We\ need\ to\ show:$$

$$\sqrt{a}+\sqrt{b}+\sqrt{c}\ge\frac{1}{2}\left(9-(a^2+b^2+c^2)\right)$$

$$2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)+(a^2+b^2+c^2)\ge 9\ or,\sum\left(\sqrt{a}+\sqrt{a}+a^2\right)\ge 9$$

$$\sum\left(\sqrt{a}+\sqrt{a}+a^2\right)\overset{AM-GM}{\ge}\sum 3a=3(a+b+c)=3\times 3=9$$

$$\frac{a^2+b^2+c^2}{abc}=\sum\frac{a}{bc}=\sum\frac{\left(\sqrt{a}\right)^2}{bc}\overset{Bergstrom}{\ge}\frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}{ab+bc+ca}\overset{Lemma}{\ge}$$

$$\ge\frac{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)}=\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)$$

$$a^2+b^2+c^2\ge abc\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)$$

Equality holds for a=b=c=1.