ROMANIAN MATHEMATICAL MAGAZINE

For $a, b, c \geq 0, ab + bc + ca = 3$ prove that:

$$\frac{1}{a+3b} + \frac{1}{b+3c} + \frac{1}{c+3a} \geq \frac{3}{4}$$

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$$\frac{1}{a+3b} + \frac{1}{b+3c} + \frac{1}{c+3a} \overset{?}{\geq} \frac{3}{4}$$

$$\Leftrightarrow 12\sum_{cyc} a^2 + 52\sum_{cyc} ab \overset{?}{\geq} 9\sum_{cyc} a^2b + 27\sum_{cyc} ab^2 + 84abc$$

$$\Leftrightarrow 12\sum_{cyc} a^2 + 3\sum_{cyc} ab + 49\sum_{cyc} ab \overset{?}{\geq}$$

$$27\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 3abc\right) + 84abc - 18\sum_{cyc} a^2b$$

$$\overset{ab+bc+ca=3}{\Leftrightarrow} 12\sum_{cyc} a^2 + 3\sum_{cyc} ab + 147 + 18\sum_{cyc} a^2b \boxed{\overset{?}{\underset{(*)}{\geq}}} 3abc + 81\sum_{cyc} a$$

$$\text{Let } F(a,b,c) = 12\sum_{cyc} a^2 + 3\sum_{cyc} ab + 18\sum_{cyc} a^2b + 147 - 3abc - 81\sum_{cyc} a$$

$$\text{and then: } F(a,a,a) = 36a^2 + 9a^2 + 54a^3 + 147 - 3a^3 - 243a$$

$$= 3(a-1)^2(17a+49) \geq 0\ (\because a \geq 0) \therefore F(a,a,a) \overset{(1)}{\geq} 0\ \forall\ a \geq 0$$

$$\text{Also, } F(a,b,0) = 12(a^2+b^2) + 3ab + 18a^2b + 147 - 81(a+b) \text{ and when:}$$

$$a = 0, F(a,b,0) = 12b^2 - 81b + 147 > 0 \because \text{discriminant} = 81^2 - 48.147$$

$$= -495 < 0 \text{ and when: } a > 0, F(a,b,0) =$$

$$12\left(a^2 + \frac{9}{a^2}\right) + 9 + 18a^2.\frac{3}{a} + 147 - 81\left(a + \frac{3}{a}\right)$$

$$(\because c = 0 \land ab + bc + ca = 3 \Rightarrow ab = 3) = \frac{3}{a^2}(4a^4 - 9a^3 + 52a^2 - 81a + 36)$$

$$= \frac{1}{729a^2}\left((9a-8)^2\left((a-26)^2 + 107a^2 + a + 552\right) + 2949a + 140\right) > 0$$

$$(\because a > 0) \therefore F(a,b,0) \overset{(2)}{\geq} 0\ \forall\ a,b \geq 0 \mid ab = 3 \text{ and so, (1) and (2)} \Rightarrow F(a,b,c) \geq 0$$

$$\forall\ a,b,c \geq 0 \mid ab + bc + ca = 3 \Rightarrow (*) \text{ is true} \therefore \frac{1}{a+3b} + \frac{1}{b+3c} + \frac{1}{c+3a} \geq \frac{3}{4}$$

$$\forall\ a,b,c \geq 0 \mid ab + bc + ca = 3, '' = '' \text{ iff } a = b = c = 1 \text{ (QED)}$$