ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0, k \geq 1$ then prove that :

$$\frac{1}{a^k(1+a^2)}+\frac{1}{b^k(1+b^2)}+\frac{1}{c^k(1+c^2)}+\frac{(abc)^{k+1}}{2}\geq 2$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1-a}{a(1+a^2)}+\frac{a^2b^2c^2-abc}{2}=\sum_{cyc}\frac{1+a^2-a^2-a}{a(1+a^2)}+\frac{a^2b^2c^2-abc}{2}$$

$$=\sum_{cyc}\frac{1}{a}-\sum_{cyc}\frac{a}{1+a^2}-\sum_{cyc}\frac{1}{1+a^2}+\frac{a^2b^2c^2-abc}{2}\overset{AM-GM}{\geq}$$

$$\sum_{cyc}\frac{1}{a}-\sum_{cyc}\frac{a}{2a}-\sum_{cyc}\frac{1}{2a}+\frac{a^2b^2c^2-abc}{2}=\frac{1}{2}\left(\sum_{cyc}\frac{1}{a}-3+a^2b^2c^2-abc\right)$$

$$\overset{AM-GM}{\geq}\frac{1}{2}\left(\frac{3}{\sqrt[3]{abc}}-3+a^2b^2c^2-abc\right)=\frac{1}{2}\left(\frac{m^7-m^4-3m+3}{m}\right)\ (\because m=\sqrt[3]{abc})$$

$$=\frac{(m-1)^2(m^5+2m^4+3m^3+3m^2+3m+3)}{2m}\geq 0$$

$$(\because a,b,c>0\Rightarrow m=\sqrt[3]{abc}>0)\ \therefore \sum_{cyc}\frac{1-a}{a(1+a^2)}+\frac{a^2b^2c^2-abc}{2}\geq 0\rightarrow (1)$$

$$\text{and } k\geq 1\Rightarrow \frac{1}{a^k(1+a^2)}+\frac{1}{b^k(1+b^2)}+\frac{1}{c^k(1+c^2)}+\frac{(abc)^{k+1}}{2}\overset{Bernoulli}{\geq}$$

$$\sum_{cyc}\left(\left(\frac{1}{1+a^2}\right)\left(1+\left(\frac{1}{a}-1\right)k\right)\right)+\left(\frac{abc}{2}\right)(1+(abc-1)k)$$

$$=\sum_{cyc}\frac{1}{1+a^2}+\frac{abc}{2}+k\left(\sum_{cyc}\frac{1-a}{a(1+a^2)}+\frac{a^2b^2c^2-abc}{2}\right)\overset{\substack{k\geq 1\\ and\\ via\ (1)}}{\geq}$$

$$\sum_{cyc}\frac{1}{1+a^2}+\frac{abc}{2}+\sum_{cyc}\frac{1-a}{a(1+a^2)}+\frac{a^2b^2c^2-abc}{2}=\sum_{cyc}\frac{1}{a(1+a^2)}+\frac{a^2b^2c^2}{2}$$

$$=\sum_{cyc}\frac{1+a^2-a^2}{a(1+a^2)}+\frac{a^2b^2c^2}{2}=\sum_{cyc}\frac{1}{a}-\sum_{cyc}\frac{a^2}{a(1+a^2)}+\frac{a^2b^2c^2}{2}\overset{AM-GM}{\geq}$$

$$\frac{3}{\sqrt[3]{abc}}-\sum_{cyc}\frac{a^2}{2a^2}+\frac{a^2b^2c^2}{2}=\frac{3}{m}-\frac{3}{2}+\frac{m^6}{2}\overset{?}{\geq}2\Leftrightarrow m^7-7m+6\overset{?}{\geq}0$$

$$\Leftrightarrow (\mathbf{m}-\mathbf{1})^{2}\left(\mathbf{m}^{5}+\mathbf{2m}^{4}+\mathbf{3m}^{3}+\mathbf{4m}^{2}+\mathbf{5m}+\mathbf{6}\right)\overset{?}{\geq}\mathbf{0}\rightarrow \mathbf{true}$$

$$\therefore \frac{\mathbf{1}}{a^{\mathbf{k}}(\mathbf{1}+a^{2})}+\frac{\mathbf{1}}{\mathbf{b}^{\mathbf{k}}(\mathbf{1}+\mathbf{b}^{2})}+\frac{\mathbf{1}}{\mathbf{c}^{\mathbf{k}}(\mathbf{1}+\mathbf{c}^{2})}+\frac{(a\mathbf{bc})^{\mathbf{k}+\mathbf{1}}}{\mathbf{2}}\geq \mathbf{2}\ \forall\ a,\mathbf{b},\mathbf{c}>\mathbf{0},\mathbf{k}\geq \mathbf{1},$$

$$''=''\ \ \mathbf{iff}\ a=\mathbf{b}=\mathbf{c}=\mathbf{1}\ (\mathbf{QED})$$