ROMANIAN MATHEMATICAL MAGAZINE

If $k \geq 0$ then prove that :

$$\prod_{cyc}\left(ka^2+(b+c)^2\right) \geq \left((k-1)(a+b+c)(ab+bc+ca)+(1-3k)abc\right)^2$$

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$$\prod_{cyc}\left(ka^2+(b+c)^2\right) \geq \left((k-1)\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)+(1-3k)abc\right)^2 \Leftrightarrow$$

$$k^3a^2b^2c^2+k^2\sum_{cyc}\left(a^2b^2(a+b)^2\right)+k\sum_{cyc}\left(a^2(a+b)^2(c+a)^2\right)+\prod_{cyc}(b+c)^2 \overset{?}{\geq}$$

$$(k^2-2k+1)\left(\sum_{cyc} a\right)^2\left(\sum_{cyc} ab\right)^2+(1-6k+9k^2)a^2b^2c^2+$$

$$(8k-6k^2-2)abc\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) \Leftrightarrow k^3a^2b^2c^2+$$

$$k^2\begin{pmatrix}\sum_{cyc}\left(a^2b^2(a+b)^2\right)-\left(\sum_{cyc} a\right)^2\left(\sum_{cyc} ab\right)^2-9a^2b^2c^2+\\ 6abc\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)\end{pmatrix}+$$

$$k\left(T=\begin{pmatrix}\sum_{cyc}\left(a^2(a+b)^2(c+a)^2\right)+2\left(\sum_{cyc} a\right)^2\left(\sum_{cyc} ab\right)^2+6a^2b^2c^2-\\ 8abc\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)\end{pmatrix}\right)+$$

$$\prod_{cyc}(b+c)^2-\left(\sum_{cyc} a\right)^2\left(\sum_{cyc} ab\right)^2-a^2b^2c^2+2abc\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)\overset{?}{\geq} 0$$

$$\Leftrightarrow k^2a^2b^2c^2-2kabc\left(\sum_{cyc} a^3+3abc+\sum_{cyc} a^2b+\sum_{cyc} ab^2\right)+T\ \boxed{\overset{?}{\underset{(*)}{\geq}}}\ 0\ (\because k \geq 0)$$

$$\left(\because \prod_{cyc}(b+c)^2 - \left(\sum_{cyc} a\right)^2\left(\sum_{cyc} ab\right)^2 - a^2b^2c^2 + 2abc\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) = 0\right)$$

and now, LHS of $(*)$ is a quadratic polynomial in $''kabc''$ with discriminant $=$

$$4\left(\sum_{cyc} a^3 + 3abc + \sum_{cyc} a^2b + \sum_{cyc} ab^2\right)^2 - 4T = 4T - 4T = 0$$

$$\left(\begin{array}{l}\because \left(\sum_{cyc} a^3 + 3abc + \sum_{cyc} a^2b + \sum_{cyc} ab^2\right)^2 = \sum_{cyc} a^6 + 2\sum_{cyc} a^5b + 2\sum_{cyc} ab^5 + 3\sum_{cyc} a^4b^2 + \\ 3\sum_{cyc} a^2b^4 + 8abc\sum_{cyc} a^3 + 4\sum_{cyc} a^3b^3 + 10abc\left(\sum_{cyc} a^2b + 2\sum_{cyc} ab^2\right) + 15a^2b^2c^2 = T\end{array}\right)$$

$$\Rightarrow \text{LHS of } (*) \geq 0 \Rightarrow (*) \text{ is true} \therefore \prod_{cyc}\left(ka^2 + (b+c)^2\right) \geq$$

$$\left((k-1)(a+b+c)(ab+bc+ca) + (1-3k)abc\right)^2 \ \forall\, k \geq 0, '' ='' \text{ iff } k = 0$$

$$\text{or } a = b = c = 0 \text{ or } k = \frac{\sum_{cyc} a^3 + 3abc + \sum_{cyc} a^2b + \sum_{cyc} ab^2}{abc}\ (abc \neq 0)\ (\text{QED})$$