ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC $0 \le k \le 5\sqrt{5}$ the following relationship holds :

$$\frac{ka+b}{a+c}+\frac{kb+c}{b+a}+\frac{kc+a}{c+b} \ge \frac{3(k+1)}{2}$$

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$$\frac{ka+b}{a+c}+\frac{kb+c}{b+a}+\frac{kc+a}{c+b} \overset{?}{\ge} \frac{3(k+1)}{2}$$

$$\Leftrightarrow \sum_{cyc}\frac{\big(k(y+z)+(z+x)\big)(2z+x+y)(2x+y+z)}{(2x+y+z)(2y+z+x)(2z+x+y)} \overset{?}{\ge} \frac{3(k+1)}{2}$$

$$(x=s-a, y=s-b, z=s-c \Rightarrow a=y+z, b=z+x, c=x+y)$$

$$\Leftrightarrow 2\sum_{cyc}x^3+(k+1)\sum_{cyc}x^2y+(1-k)\sum_{cyc}xy^2-12xyz \boxed{\overset{?}{\underset{(*)}{\ge}}} 0$$

$$\text{Let } F(X,Y,Z)=2\sum_{cYc}X^3+(k+1)\sum_{cYc}X^2Y+(1-k)\sum_{cYc}XY^2-12XYZ\ \forall\, X,Y,Z\ge 0$$

$$\text{and then : } F(1,1,1)=0 \to (1) \text{ and } F(X,Y,0)=$$

$$2(X^3+Y^3)+(k+1)X^2Y+(1-k)XY^2$$

$$=2(X^3+Y^3)+X^2Y+XY^2+k(X^2Y-XY^2) \text{ and it's trivially} \ge 0 \text{ when : } X\ge Y$$

$$\text{and when : } X<Y, F(X,Y,0)=2(X^3+Y^3)+X^2Y+XY^2-XYk(Y-X) \overset{?}{\ge} 0$$

$$\Leftrightarrow \left(2(X^3+Y^3)+X^2Y+XY^2\right)^2 \overset{?}{\ge} k^2X^2Y^2(Y-X)^2 \text{ and } \because 0\le k\le 5\sqrt{5}$$

$$\therefore \text{it suffices to prove : } \left(2(X^3+Y^3)+X^2Y+XY^2\right)^2 \overset{?}{\ge} 125X^2Y^2(Y-X)^2$$

$$\Leftrightarrow X^6+X^5Y-30X^4Y^2+65X^3Y^3-30X^2Y^4+XY^5+Y^6 \overset{?}{\ge} 0$$

$$\Leftrightarrow (X^2-3XY+Y^2)^2(X^2+7XY+Y^2) \overset{?}{\ge} 0 \to \text{true} \because X,Y\ge 0 \Rightarrow F(X,Y,0)\ge 0$$

$$\forall\, X,Y\ge 0 \to (2) \therefore (1) \text{ and } (2) \Rightarrow F(X,Y,Z)\ge 0\ \forall\, X,Y,Z\ge 0 \Rightarrow (*) \text{ is true}$$

$$(\because x,y,z>0) \therefore \frac{ka+b}{a+c}+\frac{kb+c}{b+a}+\frac{kc+a}{c+b} \ge \frac{3(k+1)}{2}\ \forall\, ABC \wedge 0\le k\le 5\sqrt{5},$$

$$'' = '' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$