ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sqrt{\frac{a}{3a+5b}}+\sqrt{\frac{b}{3b+5c}}+\sqrt{\frac{c}{3c+5a}}\le\frac{3\sqrt{2}}{4}$$

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$$\sum_{cyc}\sqrt{\frac{a}{3a+5b}}=$$

$$=\frac{1}{\sqrt{(3a+5b)(3b+5c)(3c+5a)}}\cdot\sum_{cyc}\sqrt{a(3b+5c)(3c+5a)}\le$$

$$\overset{CBS}{\le}\frac{1}{\sqrt{(3a+5b)(3b+5c)(3c+5a)}}\cdot\sqrt{\sum_{cyc}\left(a(3b+5c)\right)}\cdot\sqrt{\sum_{cyc}(3c+5a)}=$$

$$=\frac{1}{\sqrt{\prod_{cyc}(3a+5b)}}\cdot\sqrt{\left(8\sum_{cyc}ab\right)\left(8\sum_{cyc}a\right)}$$

$$=\sqrt{\frac{64\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2+3abc\right)}{45\sum_{cyc}a^2b+75\sum_{cyc}ab^2+152abc}}\overset{?}{\le}\frac{3\sqrt{2}}{4}$$

$$\Leftrightarrow 107\sum_{cyc}a^2b+168abc\overset{?}{\le}163\sum_{cyc}ab^2\Leftrightarrow 107\sum_{cyc}\left((y+z)^2(z+x)\right)+$$

$$168(y+z)(z+x)(x+y)\overset{?}{\le}163\sum_{cyc}\left((y+z)(z+x)^2\right)$$

$$(x=s-a,y=s-b,z=s-c)\Leftrightarrow 56\sum_{cyc}x^3+107\sum_{cyc}x^2y-163\sum_{cyc}xy^2\ \boxed{\underset{(*)}{\overset{?}{\ge}}}\ 0$$

$$\text{Let }F(X,Y,Z)=56\sum_{cYc}X^3+107\sum_{cYc}X^2Y-163\sum_{cYc}XY^2\ \forall X,Y,Z\ge 0\text{ and then :}$$

$$F(1,1,1)=0\to (1)\text{ and }F(X,Y,0)=56(X^3+Y^3)+107X^2Y-163XY^2$$

$$=\frac{(504X+1523Y)(9X-5Y)^2+5643XY^2+2749Y^3}{729}\ge 0\Rightarrow F(X,Y,0)\ge 0\to (2)$$

$$\therefore (1)\text{ and }(2)\Rightarrow F(X,Y,Z)\ge 0\ \forall X,Y,Z\ge 0\Rightarrow (*)\text{ is true}$$

$$\therefore\sqrt{\frac{a}{3a+5b}}+\sqrt{\frac{b}{3b+5c}}+\sqrt{\frac{c}{3c+5a}}\le\frac{3\sqrt{2}}{4}\ \forall\ ABC,$$

$$''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$