ROMANIAN MATHEMATICAL MAGAZINE

If $k \geq 0$ then prove that :

$$\prod_{cyc}(ka^2+(b-c)^2) \geq (k+1)^2\prod_{cyc}(a-b)^2$$

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$$\prod_{cyc}(ka^2+(b-c)^2) \overset{?}{\geq} (k+1)^2.\prod_{cyc}(a-b)^2$$

$$\Leftrightarrow k^3a^2b^2c^2 + k^2\sum_{cyc}(a^2b^2(a-b)^2) + k\sum_{cyc}(a^2(a-b)^2(c-a)^2) + \prod_{cyc}(b-c)^2$$

$$\overset{?}{\geq} \prod_{cyc}(a-b)^2 + k^2.\prod_{cyc}(a-b)^2 + 2k.\prod_{cyc}(a-b)^2 \Leftrightarrow$$

$$k^2a^2b^2c^2 + k\left(\sum_{cyc}(a^2b^2(a-b)^2) - \prod_{cyc}(a-b)^2\right) + \sum_{cyc}(a^2(a-b)^2(c-a)^2) -$$

$$2\prod_{cyc}(a-b)^2 \overset{?}{\geq} 0 \Leftrightarrow k^2a^2b^2c^2 + k.2abc\left(\sum_{cyc}a^3 + 3abc - \sum_{cyc}a^2b - \sum_{cyc}ab^2\right) +$$

$$\sum_{cyc}(a^2(a-b)^2(c-a)^2) - 2\prod_{cyc}(a-b)^2 \underset{(*)}{\overset{?}{\geq}} 0 \text{ and now, LHS of } (*) \text{ is a}$$

quadratic polynomial in "kabc" with discriminant =

$$4\left(\sum_{cyc}a^3 + 3abc - \sum_{cyc}a^2b - \sum_{cyc}ab^2\right)^2 -$$

$$4\left(\sum_{cyc}(a^2(a-b)^2(c-a)^2) - 2\prod_{cyc}(a-b)^2\right) = 0 \Rightarrow \text{LHS of } (*) \geq 0 \Rightarrow (*) \text{ is true}$$

$$\therefore \prod_{cyc}(ka^2+(b-c)^2) \geq (k+1)^2.\prod_{cyc}(a-b)^2 \ \forall\, k \geq 0, \text{"="} \text{ iff } k = 0 \text{ (QED)}$$