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If $a,b,c \geq 0$ and $ab+bc+ca =$ then prove that :

$$\frac{a}{\sqrt{4a+5bc}}+\frac{b}{\sqrt{4b+5ca}}+\frac{c}{\sqrt{4c+5ab}} \geq 1$$

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Since $a,b,c \geq 0 \wedge ab+bc+ca > 0$, hence only 2 cases are possible.

$\boxed{\text{Case 1}}$ Exactly 1 variable equals zero and WLOG we may assume $a = 0$ $(b,c > 0$ with $bc = 3)$ and then, LHS of main inequality becomes :

$$\sqrt{\frac{b^2}{4b}}+\sqrt{\frac{c^2}{4c}} = \frac{1}{2}\left(\sqrt{b}+\sqrt{c}\right) \overset{\text{AM-GM}}{\geq} \sqrt[4]{bc} = \sqrt[4]{3} \overset{?}{>} 1$$

$\boxed{\text{Case 2}}$ $a,b,c > 0$ and assigning $b+c = x, c+a = y, a+b = z \Rightarrow x+y-z = 2c > 0, y+z-x = 2a > 0$ and $z+x-y = 2b > 0 \Rightarrow x+y > z, y+z > x, z+x > y$ $\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $= s,R,r$ (say); $\sum_{cyc} a = s, abc = r^2s, \sum_{cyc} ab = 4Rr+r^2$,

$$\sum_{cyc} a^2 = s^2-8Rr-2r^2 \text{ and then, } \sum_{cyc}\frac{a}{\sqrt{4a+5bc}} = \sum_{cyc}\frac{a^{\frac{3}{2}}}{\sqrt{4a^2+5abc}}$$

$$\overset{\text{Radon}}{\geq} \frac{\left(\sum_{cyc} a\right)^{\frac{3}{2}}}{\sqrt{4\sum_{cyc} a^2+15abc}} \overset{?}{\geq} 1 \Leftrightarrow \left(\sum_{cyc} a\right)^3 - 15abc \overset{?}{\geq} 4\sum_{cyc} a^2$$

$$\Leftrightarrow 3\left(\left(\sum_{cyc} a\right)^3 - 15abc\right)^2 \overset{?}{\geq} 16\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2\right)^2 \quad \left(\because \sum_{cyc} ab = 3\right)$$

$$\overset{\text{via transformation}}{\Leftrightarrow} 3(s^3-15r^2s)^2 - 16(4Rr+r^2)(s^2-8Rr-2r^2)^2 \boxed{\underset{(*)}{\overset{?}{\geq}}} 0 \text{ and}$$

$\because 3(s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(*)$, it suffices to prove :

LHS of $(*) \overset{?}{\geq} 3(s^2-16Rr+5r^2)^3 \Leftrightarrow (80R-151r)s^4 - r(1280R^2-1952Rr-514r^2)s^2 + r^2(8192R^3-14592R^2r+2832Rr^2-439r^3)$ $\boxed{\underset{(**)}{\overset{?}{\geq}}}$ 0 and $\because (80R-151r)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(**)$,

it suffices to prove : LHS of $(**) \overset{?}{\geq} (80R-151r)(s^2-16Rr+5r^2)^2$

$$\Leftrightarrow (160R^2-460Rr+253r^2)s^2 \boxed{\underset{(***)}{\overset{?}{\geq}}} r(1536R^3-4608R^2r+2916Rr^2-417r^3)$$

$\boxed{\text{Case 1}}$ $160R^2-460Rr+253r^2 \geq 0$ and then : LHS of $(***) \overset{\text{Gerretsen}}{\geq}$

$$(160R^2 - 460Rr + 253r^2)(16Rr - 5r^2) \overset{?}{\geq} \text{RHS of } (***)$$

$$\Leftrightarrow 128t^3 - 444t^2 + 429t - 106 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(128t^2 - 188t + 53) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (***) \text{ is true}$$

$\boxed{Case\ 2}$ $160R^2 - 460Rr + 253r^2 < 0$ *and then* : LHS of $(***) \overset{\text{Gerretsen}}{\geq}$

$$(160R^2 - 460Rr + 253r^2)(4R^2 + 4Rr + 3r^2) \overset{?}{\geq} \text{RHS of } (***)$$

$$\Leftrightarrow 160t^4 - 684t^3 + 1065t^2 - 821t + 294 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)\left((t-2)(160t^2 - 44t + 249) + 351\right) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$\Rightarrow (***)$ is true $\therefore$ combining both *cases*, $(***) \Rightarrow (**) \Rightarrow (*)$ is true $\forall\, \Delta\, XYZ_{s,R,r}$

$$\therefore \frac{a}{\sqrt{4a + 5bc}} + \frac{b}{\sqrt{4b + 5ca}} + \frac{c}{\sqrt{4c + 5ab}} \geq 1 \ \forall\, a, b, c \geq 0 \,\middle|\, ab + bc + ca = 3,$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$