ROMANIAN MATHEMATICAL MAGAZINE

For $a, b \geq 0$ prove that :

$$\frac{1}{(a^2+a+1)^2}+\frac{1}{(b^2+b+1)^2} \leq 1+\frac{2}{(a+1)^4+(b+1)^4}$$

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$$\frac{1}{(a^2+a+1)^2}+\frac{1}{(b^2+b+1)^2}=\frac{2}{1+(a+1)^4+a^4}+\frac{2}{1+(b+1)^4+b^4} \leq$$

$$\frac{2}{1+(a+1)^4}+\frac{2}{1+(b+1)^4} \overset{?}{\leq} 1+\frac{2}{(a+1)^4+(b+1)^4}$$

$$\Leftrightarrow \frac{2}{1+(1+x)}+\frac{2}{1+(1+y)} \overset{?}{\leq} 1+\frac{2}{(x+1)+(y+1)}$$

$$\left(\begin{array}{l}\because (a+1)^4 \geq 1 \therefore \text{we can set} : (a+1)^4=1+x \text{ with } x \geq 0 \\ \text{and similarly, we can set} : (b+1)^4=1+y \text{ with } y \geq 0\end{array}\right)$$

$$\Leftrightarrow \frac{2(4+x+y)}{(2+x)(2+y)} \overset{?}{\leq} \frac{4+x+y}{2+x+y} \Leftrightarrow 4+xy+2x+2y \overset{?}{\geq} 4+2x+2y \Leftrightarrow xy \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \therefore \frac{1}{(a^2+a+1)^2}+\frac{1}{(b^2+b+1)^2} \leq 1+\frac{2}{(a+1)^4+(b+1)^4}\ \forall a, b \geq 0,$$

$$''='' \text{ iff } a=b=0 \text{ (QED)}$$