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For $a, b, c \geq 0$ prove that :

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}+\frac{1}{(1+c)^2}+\frac{abc}{4}\geq 1$$

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If $a=b=c=0$, then : $LHS-RHS=3-1>0$; if exactly 2 variables equal to zero and WLOG we may assume $b=c=0$ $(a>0)$, then : $LHS-RHS=\frac{1}{(1+a)^2}+2-1>0$ and if exactly 1 variable equals to zero and WLOG we may assume $a=0$ $(b,c>0)$, then : $LHS-RHS=1+\frac{1}{(1+b)^2}+\frac{1}{(1+c)^2}-1>0$ and we now focus on the case when : $a,b,c>0$ and then :

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}+\frac{1}{(1+c)^2}+\frac{abc}{4}\geq 1 \Leftrightarrow a^3b^3c^3+2a^2b^2c^2\left(\sum_{cyc}ab\right)+$$

$$abc\left(\sum_{cyc}a^2b^2\right)+4a^2b^2c^2\sum_{cyc}a+2abc\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)+4a^2b^2c^2+$$

$$abc\sum_{cyc}a^2+4\sum_{cyc}a^2+8\sum_{cyc}a+8\ \boxed{\overset{?}{\underset{(*)}{\geq}}}\ 4abc\sum_{cyc}ab+14abc\sum_{cyc}a+31abc$$

$$\text{and now, } 2a^2b^2c^2\left(\sum_{cyc}ab\right)+2\sum_{cyc}a^2-4abc\sum_{cyc}ab\geq 2a^2b^2c^2\left(\sum_{cyc}ab\right)+$$

$$2\sum_{cyc}ab-4abc\sum_{cyc}ab=2\left(\sum_{cyc}ab\right)(abc-1)^2\geq 0$$

$$\therefore 2a^2b^2c^2\left(\sum_{cyc}ab\right)+2\sum_{cyc}a^2\ \boxed{\overset{①}{\geq}}\ 4abc\sum_{cyc}ab$$

$$\text{We have : } abc\left(\sum_{cyc}a^2b^2\right)+4a^2b^2c^2\sum_{cyc}a+\frac{abc}{4}\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)+\frac{1}{2}\sum_{cyc}a^2$$

$$+\sum_{cyc}(a^2b^2c^2+a^3bc+a)+5\sum_{cyc}a-14abc\sum_{cyc}a\overset{AM-GM}{\geq}5a^2b^2c^2\sum_{cyc}a+$$

$$\frac{3}{2}a^2b^2c^2+\frac{1}{2}\sum_{cyc}a^2+3\sum_{cyc}a^2bc+5\sum_{cyc}a-14abc\sum_{cyc}a\overset{AM-GM}{\geq}5a^2b^2c^2\sum_{cyc}a+$$

$$\left(\frac{1}{2}\cdot\sum_{cyc}2a^2bc+3\sum_{cyc}a^2bc-14abc\sum_{cyc}a\right)+5\sum_{cyc}a$$

$$=5\left(\sum_{cyc}a\right)(a^2b^2c^2-2abc+1)=5\left(\sum_{cyc}a\right)(abc-1)^2\geq 0$$

$$\therefore abc\left(\sum_{cyc}a^2b^2\right)+4a^2b^2c^2\sum_{cyc}a+\frac{abc}{4}\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)+\frac{1}{2}\sum_{cyc}a^2+$$

$$3a^2b^2c^2+abc\sum_{cyc}a^2+6\sum_{cyc}a\ \boxed{\overset{②}{\geq}}\ 14abc\sum_{cyc}a\ \therefore ①\ \text{and}\ ② \Rightarrow \text{to prove } (*),$$

$$\text{it suffices to prove}: a^3b^3c^3+\frac{7abc}{4}\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)+a^2b^2c^2+\frac{3}{2}\sum_{cyc}a^2+$$

$$2\sum_{cyc}a+8-31abc\ \boxed{\underset{(**)}{\overset{?}{\geq}}}\ 0 \text{ and indeed, LHS of } (**) \overset{AM-GM}{\geq}$$

$$t^9+\frac{21}{2}t^6+t^6+\frac{9}{2}t^2+6t+8-31t^3\ \left(t=\sqrt[3]{abc}\right)$$

$$=\frac{(t-1)^2}{2}.\left(2t^7+4t^6+6t^5+31t^4+56t^3+81t^2+44t+16\right)\geq 0\ (\because t>0)$$

$$\Rightarrow (**)\Rightarrow(*) \text{ is true } \therefore \frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}+\frac{1}{(1+c)^2}+\frac{abc}{4}\geq 1\ \forall\, a,b,c\geq 0,$$

$$'' = ''\ \text{iff } a=b=c=1\ (QED)$$