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For $a, \mathrm{b}, \mathrm{c} \geq 0$ prove that :

$$\frac{1}{1+a}+\frac{1}{1+\mathrm{b}}+\frac{1}{1+\mathrm{c}} \geq \frac{144}{96+4(a+\mathrm{b}+\mathrm{c})+3a\mathrm{bc}}$$

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If $a=\mathrm{b}=\mathrm{c}=0$, then : $\mathrm{LHS}-\mathrm{RHS}=3-\dfrac{144}{96}>0$; if exactly 2 variables equal to zero and WLOG we may assume $\mathrm{b}=\mathrm{c}=0$ $(a>0)$, then : $\mathrm{LHS}-\mathrm{RHS}=$ $\dfrac{1}{1+a}+2-\dfrac{144}{96+4a}=\dfrac{1}{1+a}+\dfrac{12+2a}{24+a}>0$ and if exactly 1 variable equals to zero and WLOG we may assume $a=0$ $(\mathrm{b},\mathrm{c}>0)$, then : $\mathrm{LHS}-\mathrm{RHS}=$

$$1+\frac{1}{1+\mathrm{b}}+\frac{1}{1+\mathrm{c}}-\frac{36}{24+\mathrm{b}+\mathrm{c}}=\frac{\mathrm{b}^2\mathrm{c}+\mathrm{bc}^2+15\mathrm{b}+15\mathrm{c}+36+2\mathrm{b}^2+2\mathrm{c}^2-8\mathrm{bc}}{(1+\mathrm{b})(1+\mathrm{c})(24+\mathrm{b}+\mathrm{c})}$$

$>0 \because \mathrm{b}^2\mathrm{c}+\mathrm{bc}^2+4\mathrm{b}+4\mathrm{c} \overset{\mathrm{AM-GM}}{\geq} 4.\sqrt[4]{16\mathrm{b}^4\mathrm{c}^4}=8\mathrm{bc}$ and we now focus on the case when $a,\mathrm{b},\mathrm{c}>0$ & then : $\dfrac{1}{1+a}+\dfrac{1}{1+\mathrm{b}}+\dfrac{1}{1+\mathrm{c}} \geq \dfrac{144}{96+4(a+\mathrm{b}+\mathrm{c})+3a\mathrm{bc}}$

$$\Leftrightarrow 3a\mathrm{bc}\sum_{\mathrm{cyc}} a\mathrm{b}+6a\mathrm{bc}\sum_{\mathrm{cyc}} a+4\sum_{\mathrm{cyc}} a^2\mathrm{b}+4\sum_{\mathrm{cyc}} a\mathrm{b}^2+8\sum_{\mathrm{cyc}} a^2+60\sum_{\mathrm{cyc}} a+144$$

$$\boxed{\overset{?}{\underset{(*)}{\geq}}}\ 123a\mathrm{bc}+32\sum_{\mathrm{cyc}} a\mathrm{b} \text{ \& now, } 4\sum_{\mathrm{cyc}}(a^2\mathrm{b}+a\mathrm{b}^2+4a+4\mathrm{b}) \overset{\mathrm{AM-GM}}{\geq}$$

$16\displaystyle\sum_{\mathrm{cyc}}\sqrt[4]{16a^4\mathrm{b}^4}=32\sum_{\mathrm{cyc}} a\mathrm{b}$ and so, in order to prove $(*)$, it suffices to prove :

$$3a\mathrm{bc}\sum_{\mathrm{cyc}} a\mathrm{b}+6a\mathrm{bc}\sum_{\mathrm{cyc}} a+8\sum_{\mathrm{cyc}} a^2+28\sum_{\mathrm{cyc}} a+144\ \boxed{\overset{?}{\underset{(**)}{\geq}}}\ 123a\mathrm{bc}$$

We have via AM $-$ GM, LHS of $(**)$ $-$ RHS of $(**) \geq$

$9\mathrm{t}^5+18\mathrm{t}^4+24\mathrm{t}^2+84\mathrm{t}+144-123\mathrm{t}^3\ \left(\mathrm{t}=\sqrt[3]{a\mathrm{bc}}\right)$

$=3(\mathrm{t}-2)^2(3\mathrm{t}^3+18\mathrm{t}^2+19\mathrm{t}+12) \geq 0\ (\because \mathrm{t}>0) \Rightarrow (**) \Rightarrow (*)$ is true

$$\therefore \frac{1}{1+a}+\frac{1}{1+\mathrm{b}}+\frac{1}{1+\mathrm{c}} \geq \frac{144}{96+4(a+\mathrm{b}+\mathrm{c})+3a\mathrm{bc}}\ \forall\, a,\mathrm{b},\mathrm{c} \geq 0,$$

$''=''$ iff $a=\mathrm{b}=\mathrm{c}=2$ (QED)