ROMANIAN MATHEMATICAL MAGAZINE

For $x, y \in \left(0, \frac{\pi}{2}\right]$ prove that :

$$\frac{1}{\sin^2 x + \sin^2 y} + \frac{1}{\sin x} + \frac{1}{\sin y} \le 2 + \frac{1}{\sin x \sin y}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Since $x, y \in \left(0, \frac{\pi}{2}\right]$ $\therefore 0 < \sin x, \sin y \le 1 \Rightarrow \frac{1}{\sin x}, \frac{1}{\sin y} \ge 1$ *and* so we *can assign* : $\frac{1}{\sin x} = 1 + a$ *and* $\frac{1}{\sin y} = 1 + b$; where $a, b \ge 0$ *and* then :

$$\frac{1}{\sin^2 x + \sin^2 y} + \frac{1}{\sin x} + \frac{1}{\sin y} \overset{?}{\le} 2 + \frac{1}{\sin x . \sin y}$$

$$\Leftrightarrow \frac{1}{\left(\frac{1}{1+a}\right)^2 + \left(\frac{1}{1+b}\right)^2} + 1 + a + 1 + b \overset{?}{\le} 2 + (1+a)(1+b)$$

$$\Leftrightarrow \frac{(a+1)^2(b+1)^2}{(a+1)^2 + (b+1)^2} \overset{?}{\le} ab + 1 \Leftrightarrow a^3b + ab^3 - a^2b^2 - 2ab + 1 \underset{(*)}{\underbrace{\overset{?}{\ge}}} 0$$

Now, $a^3b + ab^3 - a^2b^2 - 2ab + 1 = ab(a^2 + b^2) - a^2b^2 - 2ab + 1 \ge ab(2ab) - a^2b^2 - 2ab + 1$ ($\because a^2 + b^2 \ge 2ab \;\forall a, b \in \mathbb{R}$ *and* $\because a, b \ge 0 \Rightarrow ab \ge 0$) $= a^2b^2 - 2ab + 1 = (ab - 1)^2 \ge 0 \Rightarrow (*)$ is true with equ*al*ity iff $a = b \wedge ab = 1$ $\Rightarrow$ iff $a = b = 1 \Rightarrow$ iff $\frac{1}{\sin x} = \frac{1}{\sin y} = 2 \Rightarrow$ iff $x = y = \frac{\pi}{6}$ $\left(\because x, y \in \left(0, \frac{\pi}{2}\right]\right)$ *and* so,

$$\frac{1}{\sin^2 x + \sin^2 y} + \frac{1}{\sin x} + \frac{1}{\sin y} \le 2 + \frac{1}{\sin x . \sin y} \;\forall x, y \in \left(0, \frac{\pi}{2}\right],$$

$'' = ''$ iff $x = y = \frac{\pi}{6}$ (QED)