ROMANIAN MATHEMATICAL MAGAZINE

Let x, y, z be the angles of acute triangle. Prove that:

$$\frac{x\tan^2 x}{\ln\left(\frac{1}{\cos x}\right)}+\frac{x\tan^2 x}{\ln\left(\frac{1}{\cos x}\right)}+\frac{x\tan^2 x}{\ln\left(\frac{1}{\cos x}\right)}>2\tan x\cdot\tan y\cdot\tan z$$

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$$\text{Let } f(x)= x\tan x-2\ln\left(\frac{1}{\cos x}\right)=x\tan x-2\ln\sec x\ ,x\in\left(0,\frac{\pi}{2}\right)$$

$$f'(x)=x\sec^2 x-\tan x$$

$$\text{Let } g(x)=x\sec^2 x-\tan x\ \text{ then } g'(x)=2x\sec^2 x\tan x>0 \text{ as } x\in\left(0,\frac{\pi}{2}\right)$$

$$\text{so } g(x) \text{ is increasing and } g(0)=0 \text{ so } g(x)>g(0) \text{ or } x\sec^2 x-\tan x>0$$

$$\text{From this result we can say } f'(x)=x\sec^2 x-\tan x>0 \text{ so } f(x) \text{ increasing}$$

$$\text{and } f(0)=0 \text{ so } f(x)>f(0) \text{ or, } x\tan x-2\ln\left(\frac{1}{\cos x}\right)>0 \text{ or } \frac{x\tan x}{\ln\left(\frac{1}{\cos x}\right)}>2\ (1)$$

$$\text{We know that in any triangle } \sum\tan A=\tan A.\tan B.\tan C\ (2)$$

$$\frac{x\tan^2 x}{\ln\left(\frac{1}{\cos x}\right)}+\frac{x\tan^2 x}{\ln\left(\frac{1}{\cos x}\right)}+\frac{x\tan^2 x}{\ln\left(\frac{1}{\cos x}\right)}=\sum\frac{x\tan^2 x}{\ln\left(\frac{1}{\cos x}\right)}=$$

$$=\sum\frac{x\tan^2 x}{\ln\left(\frac{1}{\cos x}\right)}\overset{(1)}{>}\sum 2\tan x\overset{2}{=}\ 2\tan x\cdot\tan y\cdot\tan z$$