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1501. If $a, b, c > 0, a^3 + b^3 + c^3 = 3$ and $\lambda \geq 0$ with $n \in \mathbb{N}$, then :

$$\sum_{cyc} \frac{a^{3n+2}}{b^2 + \lambda} \geq \frac{3}{\lambda + 1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Firstly, } a^3 + 2b^3 \overset{A-G}{\geq} 3ab^2, b^3 + 2c^3 \overset{A-G}{\geq} 3bc^2 \text{ and } c^3 + 2a^3 \overset{A-G}{\geq} 3ca^2$$

$$\therefore 3\sum_{cyc} a^3 \geq 3\sum_{cyc} ab^2 \Rightarrow \sum_{cyc} ab^2 \leq \sum_{cyc} a^3 \to (1)$$

$$\text{Now, } \sum_{cyc} \frac{a^{3n+2}}{b^2 + \lambda} = \sum_{cyc} \frac{a^{3n+3}}{ab^2 + \lambda a} = \sum_{cyc} \frac{(a^3)^{n+1}}{ab^2 + \lambda a} \overset{\text{Holder}}{\geq} \frac{\left(\sum_{cyc} a^3\right)^{n+1}}{3^{n-1}\left(\sum_{cyc} ab^2 + \lambda \sum_{cyc} a\right)} \geq$$

$$\overset{\text{via (1)}}{\geq} \frac{\left(\sum_{cyc} a^3\right)^{n+1}}{3^{n-1}\left(\sum_{cyc} a^3 + 3\lambda\right)} \left(\because 3 = \sum_{cyc} a^3 \overset{\text{Holder}}{\geq} \frac{1}{9}\left(\sum_{cyc} a\right)^3 \Rightarrow \sum_{cyc} a \leq 3\right) =$$

$$\overset{a^3+b^3+c^3=3}{=} \frac{(3)^{n+1}}{3^{n-1}(3+3\lambda)} \therefore \sum_{cyc} \frac{a^{3n+2}}{b^2 + \lambda} \geq \frac{3}{\lambda + 1} \ \forall a, b, c > 0 \,\middle|\, a^3 + b^3 + c^3 = 3$$

$$\text{and } \lambda \geq 0 \text{ with } n \in \mathbb{N}, \text{''} = \text{''} \text{ iff } a = b = c = 1 \text{ (QED)}$$

1502. If $a, b, c > 0, a + b + c = abc, \lambda \geq 0$ then:

$$\frac{1}{\lambda + ab} + \frac{1}{\lambda + bc} + \frac{1}{\lambda + ca} \leq \frac{3}{\lambda + 3}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\text{Let } a = \frac{1}{x}, b = \frac{1}{y}, c = \frac{1}{z} \text{ then } a + b + c = abc \text{ or}$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{xyz} \text{ or}, xy + yz + zx = 1 \ (1)$$

$$\frac{1}{\lambda + ab} + \frac{1}{\lambda + bc} + \frac{1}{\lambda + ca} = \sum \frac{1}{\lambda + ab} = \sum \frac{1}{\lambda + \frac{1}{x} \cdot \frac{1}{y}} =$$

$$=\sum\frac{xy}{\lambda xy+1}=\frac{1}{\lambda}\sum\left(1-\frac{1}{\lambda xy+1}\right)=$$

$$=\frac{3}{\lambda}-\frac{1}{\lambda}\sum\frac{1^2}{\lambda xy+1}\overset{Bergstrom}{\leq}\frac{3}{\lambda}-\frac{1}{\lambda}\frac{(1+1+1)^2}{\lambda(xy+yz+zx)+3}\overset{(1)}{=}$$

$$=\frac{3}{\lambda}-\frac{1}{\lambda}\frac{9}{\lambda+3}=\frac{1}{\lambda}\left(3-\frac{9}{\lambda+3}\right)=\frac{1}{\lambda}\frac{3\lambda}{\lambda+3}=\frac{3}{\lambda+3}$$

$$Equality\ holds\ for\ a=b=c=\sqrt{3}$$

1503. **If $a,b,c\geq 0, a+b+c=2$ and $\lambda\geq 1$, then:**

$$\sum_{cyc}\sqrt{\frac{b+c}{a^2+\lambda}}\geq 2\sqrt{\frac{2}{\lambda}}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have :

$$\sum_{cyc}\sqrt{\frac{b+c}{a^2+\lambda}}=\sum_{cyc}\frac{b+c}{\sqrt{(a^2+\lambda)(2-a)}}=\sum_{cyc}\frac{b+c}{\sqrt{2\lambda-a[(a-1)^2+(\lambda-1)]}}\geq$$

$$\geq\sum_{cyc}\frac{b+c}{\sqrt{2\lambda}}\geq(a+b+c)\sqrt{\frac{2}{\lambda}}=2\sqrt{\frac{2}{\lambda}}$$

as desired. Equality holds iff $a=2, b=c=0$ and permutation.

1504. **If $a,b,c>0$ then:**

$$\sum\frac{a+b}{a^5+b^5+8}\leq\frac{3}{5}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$We\ will\ show\ a^5+b^5+8\geq 5(a+b)\ (1)$$

$$Proof: we\ need\ to\ show\ a^5+b^5+8\geq 5(a+b)\ or$$

$$\frac{(a+b)^5}{16}+8\ge 5(a+b)\ (CBS)\ \ or$$
$$(a+b)^5+128\ge 80(a+b)or$$

$$t^5-80t+128\overset{a+b=t>0}{\ge}$$

$$(t-2)^2(t^3+4t^2+12t+32)\ge 0\ true\ as\ t>0$$

$$\sum\frac{a+b}{a^5+b^5+8}\overset{(1)}{\le}\sum\frac{a+b}{5(a+b)}=\sum\frac{1}{5}=\frac{3}{5}$$

$$Equality\ holds\ for\ a=b=c$$

1505. If $x,y,z,n>0$ and $\lambda>0$ then:

$$\sum\frac{x^2+n^2}{y+\lambda z}\ge\frac{6n}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\frac{x^2+n^2}{y+\lambda z}=\sum\frac{x^2}{y+\lambda z}+\sum\frac{n^2}{y+\lambda z}\overset{Bergstrom}{\ge}$$

$$\ge\frac{(\sum x)^2}{(\sum x)(\lambda+1)}+n^2\frac{(1+1+1)^2}{(\sum x)(\lambda+1)}=$$

$$=\frac{(\sum x)^2}{(\sum x)(\lambda+1)}+n^2\frac{9}{(\sum x)(\lambda+1)}\overset{Am-Gm}{\ge}2\sqrt{\frac{(\sum x)^2}{(\sum x)(\lambda+1)}.n^2\frac{9}{(\sum x)(\lambda+1)}}=\frac{6n}{\lambda+1}$$

Equality holds for $x=y=z=n$.

1506. If $a\ge -1, \mathrm{b}\ge -1$ and $a^2+\mathrm{b}^2\ge 5$, then prove that :

$$a^3+\mathrm{b}^3\ge 7$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$a \ge -1, b \ge -1 \Rightarrow (a+1)(b+1) \ge 0 \Rightarrow a+b+ab+1 \ge 0$$
$$\Rightarrow ab \ge -a-b-1 \therefore a^2+b^2 \ge 5 \Rightarrow (a+b)^2 \ge 5+2ab \ge 5-2a-2b-2$$
$$\Rightarrow (a+b)^2+2(a+b)-3 \ge 0 \Rightarrow (a+b-1)(a+b+3) \ge 0 \Rightarrow \boxed{a+b \ge 1}$$
$$\to (1)\ (\because a \ge -1, b \ge -1 \Rightarrow a+b \ge -2 \Rightarrow a+b \not\le -3)$$
$$\boxed{\text{Case 1}}\ b \ge 2 \text{ and via } (1), a \ge 1-b \Rightarrow a^3 \ge (1-b)^3 = 1-3b+3b^2-b^3$$
$$\Rightarrow a^3+b^3-7 \ge 3b^2-3b-6 = 3(b-2)(b+1) \overset{b\ge 2}{\ge} 0 \Rightarrow a^3+b^3 \ge 7$$
$$\boxed{\text{Case 2}}\ b < 2 \text{ and } \because a^2+b^2 \ge 5 \therefore 4 > b^2 \ge 5-a^2 \Rightarrow a^2 > 1 \Rightarrow \boxed{a > 1}$$
$$(\because a \ge -1 \Rightarrow a \not< -1)$$
$$\text{Now}, a^2+b^2 \ge 5 \Rightarrow a^2 \ge 5-b^2 \Rightarrow a \ge \sqrt{5-b^2}$$
$$(\because a > 1 > 0 \text{ and } 5-b^2 > 4-b^2 > 0) \Rightarrow a^3 \ge (5-b^2)^{\frac{3}{2}}$$
$$\Rightarrow \boxed{a^3+b^3 \ge (5-b^2)^{\frac{3}{2}}+b^3 \overset{?}{>} 7} \Leftrightarrow (5-b^2)^{\frac{3}{2}} \overset{?}{>} 7-b^3,$$

which is trivially true if $b \ge \sqrt[3]{7}$ and so, we now focus on the scenario when :

$$b < \sqrt[3]{7} \text{ and then} : (5-b^2)^{\frac{3}{2}} \overset{?}{>} 7-b^3 \Leftrightarrow (5-b^2)^3 \overset{?}{>} (7-b^3)^2$$
$$\Leftrightarrow 8(2b^6-15b^4-14b^3+75b^2-76) \overset{?}{<} 0$$
$$\Leftrightarrow (b-2)(b+1)((2b-3)^2(4(b+2)^2+5)+44(2-b)+27) \overset{?}{<} 0 \to \text{true} \because$$
$$1-\le b < 2 \therefore a^3+b^3 > 7 \text{ and combining both cases}, a^3+b^3 \ge 7$$
$$\forall a \ge -1, b \ge -1 \,\big|\, a^2+b^2 \ge 5,$$
$$'' ='' \text{ iff } (a=2, b=-1) \text{ or } (a=-1, b=2) \text{ (QED)}$$

1507. If $x, y, z > 0$ and $x^2+y^2-2z^2+2xy+yz+zx \le 0$, then prove that

$$\frac{x^4+y^4}{z^4}+\frac{y^4+z^4}{x^4}+\frac{z^4+x^4}{y^4} \ge \frac{273}{8}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $x^2+y^2-2z^2+2xy+yz+zx = (x+y-z)(x+y+2z)$, then $z \ge x+y$.

$$\frac{x^4+y^4}{z^4}+\frac{y^4+z^4}{x^4}+\frac{z^4+x^4}{y^4} = \frac{x^4+y^4}{z^4}+\frac{z^4}{256}\left(\frac{1}{x^4}+\frac{1}{y^4}\right)+\frac{255z^4}{256}\left(\frac{1}{x^4}+\frac{1}{y^4}\right)+\frac{y^4}{x^4}+\frac{x^4}{y^4}$$

$$\overset{AM-GM}{\ge} 2\sqrt{\frac{x^4+y^4}{256}\left(\frac{1}{x^4}+\frac{1}{y^4}\right)}+\frac{255(x+y)^4}{256}\left(\frac{1}{x^4}+\frac{1}{y^4}\right)+2$$

$$\overset{AM-GM}{\geq} 2\sqrt{\frac{2x^2y^2}{256}\cdot\frac{2}{x^2y^2}}+\frac{255(2\sqrt{xy})^4}{256}\cdot\frac{2}{x^2y^2}+2=\frac{273}{8}.$$

Equality holds iff $x = y = \frac{z}{2}$.

1508. If $a, b, c > 0$ and $ab + bc + ca \leq 3abc$ then prove that:

$$\sqrt{\frac{a^2+b^2}{a+b}}+\sqrt{\frac{b^2+c^2}{b+c}}+\sqrt{\frac{c^2+a^2}{c+a}}+3\leq\sqrt{2}\left(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\right)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$\sqrt{2}.\sum_{cyc}\sqrt{b+c}=\sum_{cyc}\sqrt{(1+1)\left(\frac{b^2+c^2}{b+c}+\frac{2bc}{b+c}\right)}\geq\sum_{cyc}\left(\sqrt{\frac{b^2+c^2}{b+c}}+\sqrt{\frac{2bc}{b+c}}\right)$$

$$=\sum_{cyc}\sqrt{\frac{b^2+c^2}{b+c}}+\sum_{cyc}\sqrt{\frac{2}{\frac{1}{b}+\frac{1}{c}}}\overset{Jensen}{\geq}\sum_{cyc}\sqrt{\frac{b^2+c^2}{b+c}}+3\sqrt{\frac{2.3}{\sum_{cyc}\left(\frac{1}{b}+\frac{1}{c}\right)}}$$

$$=\sum_{cyc}\sqrt{\frac{b^2+c^2}{b+c}}+3\sqrt{\frac{3abc}{ab+bc+ca}}\geq\sum_{cyc}\sqrt{\frac{b^2+c^2}{b+c}}+3,$$

as desired. Equality holds iff $a = b = c = 1$.

1509. Let $a, b, c \geq 0, ab + bc + ca > 0$. Prove that:

$$\frac{b+c}{ab+ac+2bc}+\frac{c+a}{bc+ba+2ca}+\frac{a+b}{ca+cb+2ab}\leq\frac{3}{2}\cdot\frac{a+b+c}{ab+bc+ca}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$2(ab+bc+ca)\sum_{cyc}\frac{b+c}{ab+ac+2bc}=\sum_{cyc}\left(b+c+\frac{a(b+c)^2}{ab+ac+2bc}\right)$$

$$\overset{CBS}{\leq} 2\sum_{cyc}a+\sum_{cyc}a\left(\frac{b^2}{ab+bc}+\frac{c^2}{ac+bc}\right)=2\sum_{cyc}a+\sum_{cyc}\left(\frac{ab}{a+c}+\frac{ca}{a+b}\right)$$

$$=2\sum_{cyc}a+\sum_{cyc}\left(\frac{ca}{c+b}+\frac{ab}{b+c}\right)=2\sum_{cyc}a+\sum_{cyc}a=3(a+b+c),$$

as desired. Equality holds iff $(a=b=c>0)$ and

$(a=0,b,c>0)$ and permutation.

1510. If $x,y,z>0, x+y+z=1$, then :

$$\prod_{cyc}(x+2y)\leq\frac{2}{3}+3\sum_{cyc}xy^2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\prod_{cyc}(x+2y)\leq\frac{2}{3}+3\sum_{cyc}xy^2\overset{\because x+y+z=1}{\Leftrightarrow}$$

$$3\left(2\sum_{cyc}x^2y+4\sum_{cyc}xy^2+9xyz\right)\leq 2\left(\sum_{cyc}x\right)^3+9\sum_{cyc}xy^2$$

$$\Leftrightarrow 3\left(2\sum_{cyc}x^2y+4\sum_{cyc}xy^2+9xyz\right)\leq$$

$$2\left(\sum_{cyc}x^3+3\left(2xyz+\sum_{cyc}x^2y+\sum_{cyc}xy^2\right)\right)+9\sum_{cyc}xy^2$$

$$\Leftrightarrow 2\sum_{cyc}x^3+3\sum_{cyc}xy^2\geq 15xyz\to\text{true}\because\sum_{cyc}x^3\overset{A-G}{\geq}3xyz\text{ and }\sum_{cyc}xy^2\overset{A-G}{\geq}3xyz$$

$$\therefore\prod_{cyc}(x+2y)\leq\frac{2}{3}+3\sum_{cyc}xy^2\ \forall\, x,y,z>0\,\middle|\, x+y+z=1,$$

$$'' ='' \text{ iff } x = y = z = \frac{1}{3} \text{ (QED)}$$

1511. If $a, b > 0$ and $ab(a^4 + b^4) \geq 2$, then prove that :

$$a^5 + b^5 \geq 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{We shall prove that : } (a^5 + b^5)^6 \geq 2a^5b^5(a^4 + b^4)^5 \rightarrow (1)$$

$$\text{and } \because a^5 + b^5 \overset{\text{Chebyshev}}{\geq} \frac{1}{2}(a + b)(a^4 + b^4)$$

$$\therefore (a^5 + b^5)^6 \geq \frac{1}{32}(a^5 + b^5)(a + b)^5(a^4 + b^4)^5$$

$$\overset{\text{Holder}}{\geq} \frac{1}{32} \cdot \frac{1}{16} \cdot (a + b)^5(a + b)^5(a^4 + b^4)^5 \overset{?}{\geq} 2a^5b^5(a^4 + b^4)^5$$

$$\Leftrightarrow (a + b)^{10} \overset{?}{\geq} 1024a^5b^5 \Leftrightarrow (a + b)^2 \overset{?}{\geq} 4ab \rightarrow \text{true via } A - G \therefore (1) \text{ is true}$$

$$\Rightarrow (a^5 + b^5)^6 \overset{ab(a^4+b^4) \geq 2}{\geq} 64 \Rightarrow a^5 + b^5 \geq 2$$

$$\forall\, a, b > 0 \,\big|\, ab(a^4 + b^4) \geq 2, '' ='' \text{ iff } a = b = 1 \text{ (QED)}$$

1512. If $a, b > 0$ and $ab = 4$, then prove that :

$$\frac{1}{\sqrt{a^3 + 1}} + \frac{1}{\sqrt{b^3 + 1}} \geq \frac{2}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } x = \frac{1}{a}, y = \frac{1}{b} \text{ and then : } xy = \frac{1}{4} \text{ and } t = x + y \overset{A-G}{\geq} 2\sqrt{xy} = 2.\sqrt{\frac{1}{4}}$$

$$\Rightarrow t \geq 1 \rightarrow (1) \text{ and now, } \frac{1}{\sqrt{a^3 + 1}} + \frac{1}{\sqrt{b^3 + 1}} = \frac{1}{\sqrt{\frac{1}{x^3} + 1}} + \frac{1}{\sqrt{\frac{1}{y^3} + 1}}$$

$$= \frac{x.\sqrt{x}.\sqrt{x}}{\sqrt{x(x + 1)}.\sqrt{x^2 - x + 1}} + \frac{y.\sqrt{y}.\sqrt{y}}{\sqrt{y(y + 1)}.\sqrt{y^2 - y + 1}} \overset{\text{Bergstrom}}{\geq}$$

$$\frac{(x+y)^2}{\sqrt{x^2+y^2+x+y}.\sqrt{x^2+y^2-(x+y)+2}}=\frac{t^2}{\sqrt{t^2-2xy+t}.\sqrt{t^2-2xy-t+2}}$$

$$\overset{xy=\frac{1}{4}}{=}\frac{2t^2}{\sqrt{(2t^2+2t-1)(2t^2-2t+3)}}\overset{?}{\geq}\frac{2}{3}\Leftrightarrow 5t^4-8t+3\overset{?}{\geq}0$$

$$\Leftrightarrow (t-1)(5t^3+5t^2+5(t-1)+2)\overset{?}{\geq}0\rightarrow \text{true} \because t\geq 1 \text{ via } (1)$$

$$\therefore \frac{1}{\sqrt{a^3+1}}+\frac{1}{\sqrt{b^3+1}}\geq\frac{2}{3}\ \forall a,b>0\,\big|\,ab=4,''='' \text{ iff } a=b=2 \text{ (QED)}$$

1513. If $x,y>0,\ 3^x+3^y=6$ then:

$$(2x^2+y)(2y^2+x)+9xy\leq 18$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$3^x+3^y=6 \text{ or } \frac{3^x+3^y}{2}=3 \text{ or } \sqrt{3^{x+y}}\overset{AM-GM}{\leq} 3 \text{ or } 3^{\frac{x+y}{2}}\leq 3 \text{ or}$$

$$\frac{x+y}{2}\leq 1 \text{ or } x+y\leq 2 \qquad (1)$$

$$4x^2y^2=4xy.(xy)\overset{AM-GM}{\leq}4xy\frac{(x+y)^2}{4}=xy(x+y)^2\overset{(1)}{\leq}4xy \qquad (2)$$

$$x^3+y^3=(x+y)(x^2+y^2-xy)\overset{(1)}{\leq}2(x^2+y^2-xy) \qquad (3)$$

$$(2x^2+y)(2y^2+x)+9xy=4x^2y^2+2(x^3+y^3)+10xy\overset{(2)\&(3)}{\leq}$$

$$\leq 4xy+\ 4(x^2+y^2-xy)+6xy=$$

$$=4(x+y)^2+2xy\overset{(1)\&\ AM-GM}{\leq}4\cdot 2^2+\frac{2(x+y)^2}{4}\overset{(1)}{\leq}16+\frac{2^2}{2}=18$$

Equality holds for x=y=1.

1514. If $a,b,c>0$ *then prove that* :

$$\left(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}\right).\sqrt[3]{a^3+2b^3+c^3}\leq\sqrt{2}\left(\sqrt[3]{a^3+b^3}^2+\sqrt[3]{b^3+c^3}^2\right)$$

Proposed by Pavlos Trifon-Greece

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Power}-\text{Mean inequality}\Rightarrow\left(\frac{a^3+b^3}{2}\right)^{\frac{1}{3}}\geq\left(\frac{a^2+b^2}{2}\right)^{\frac{1}{2}}$$

$$\Rightarrow \sqrt{\frac{a^2+b^2}{2}} \le \left(\frac{a^3+b^3}{2}\right)^{\frac{1}{3}} \text{ and similarly}, \sqrt{\frac{b^2+c^2}{2}} \le \left(\frac{b^3+c^3}{2}\right)^{\frac{1}{3}}$$

$$\Rightarrow \left(\sqrt{\frac{a^2+b^2}{2}}+\sqrt{\frac{b^2+c^2}{2}}\right).\sqrt[3]{(a^3+b^3)+(b^3+c^3)}$$

$$\le (x+y).\sqrt[3]{2x^3+2y^3}\ \left(x=\left(\frac{a^3+b^3}{2}\right)^{\frac{1}{3}}, y=\left(\frac{b^3+c^3}{2}\right)^{\frac{1}{3}}\right)$$

$$\overset{?}{\le} \sqrt[3]{a^3+b^3}^2+\sqrt[3]{b^3+c^3}^2 = 2^{\frac{2}{3}}.(x^2+y^2) \Leftrightarrow 4(x^2+y^2)^3 \overset{?}{\ge} 2(x^3+y^3)(x+y)^3$$

$$\Leftrightarrow 2(t^2+1)^3 \overset{?}{\ge} (t^3+1)(t+1)^3\ \left(t=\frac{x}{y}\right)$$

$$\Leftrightarrow t^6-3t^5+3t^4-2t^3+3t^2-3t+1 \overset{?}{\ge} 0 \Leftrightarrow (t^2+t+1)(t-1)^4 \overset{?}{\ge} 0 \to \text{true}$$

$$\therefore \left(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}\right).\sqrt[3]{a^3+2b^3+c^3} \le \sqrt{2}\left(\sqrt[3]{a^3+b^3}^2+\sqrt[3]{b^3+c^3}^2\right)$$

$$\forall\, a,b,c>0,''='' \text{ iff } a=b=c\ (\text{QED})$$

1515. **Let $a,b,c \ge 0, a+b+c=2$. Find the minimum value of**

$$P=\sqrt{a^2+b^2+7c}+\sqrt{b^2+c^2+7a}+\sqrt{c^2+a^2+7b}.$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We will prove that the minimum of P is $2\sqrt{5}$, achieved at

$a=b=\frac{2}{3}$ and when one of a,b,c is 0 and the others are 1. Let $p:=a+b+c=2$,

$q:=ab+bc+ca,\ \ r:=abc.$ By Hölder's inequality, we have

$$\left(\sum_{cyc}\sqrt{b^2+c^2+7a}\right)^2.\sum_{cyc}\frac{(a+1)^3}{b^2+c^2+7a} \ge \left(\sum_{cyc}(a+1)\right)^3 = 125,$$

$$\sum_{cyc}\frac{(a+1)^3}{b^2+c^2+7a} = \frac{544-221q+q^2-4q^3+(768-123q+10q^2)r+32r^2}{224-84q-10q^2-2q^3+(313-27q)r-r^2}.$$

$$P^2 \geq 125.\frac{224-84q-10q^2-2q^3+(313-27q)r-r^2}{544-221q+q^2-4q^3+(768-123q+10q^2)r+32r^2} \overset{?}{\geq} (2\sqrt{5})^2$$

$$\Leftrightarrow\ f(r)=32+22q-52q^2-2q^3+(29+111q-20q^2)r-69r^2 \geq 0.$$

We have $q \leq \frac{p^2}{3}$ and $pq \geq 9r$, then $q \leq \frac{4}{3},\ \ r \leq \frac{2q}{9}$, and

$$f'(r)=29+111q-20q^2-138r > 29+111q-20q^2-138.\frac{2q}{9} > 0,$$

so f is increasing, and if $q \leq 1$, we have

$$f(r) \geq f(0)=32+22q-52q^2-2q^3=(1-q)(32+54q+2q^2) \geq 0.$$

If $1 \leq q \leq \frac{4}{3}$, we have by Schur's inequality, $r \geq \frac{p(4q-p^2)}{9}=\frac{8(q-1)}{9}$, and

$$f(r) \geq f\left(\frac{8(q-1)}{9}\right)=\frac{2}{9}(q-1)\left(\frac{4}{3}-q\right)(163+89q) \geq 0.$$

which completes the proof.

1516. Let $a,b,c \geq 0, ab+bc+ca=1$. Maximize the following expression :

$$P=\frac{1}{a+2ab+3}+\frac{1}{b+2bc+3}+\frac{1}{c+2ca+3}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We will prove that the maximum value of P is $\frac{16}{21}$. We have :

$$P \leq \frac{16}{21} \Leftrightarrow \sum_{cyc}\frac{1}{a+2ab+3} \leq \frac{16}{21} \Leftrightarrow$$

$$21\sum_{cyc}(b+2bc+3)(c+2ca+3) \leq 16\prod_{cyc}(a+2ab+3)$$

$$\Leftrightarrow a+b+c+3(ab^2+bc^2+ca^2)+\frac{121}{9}abc+\frac{70}{9}abc(a+b+c)+\frac{64}{9}(abc)^2 \geq 4.\ (1)$$

By AM − GM inequality, we have

$$16ab^2+a(2-c)^2\ge 8ab(2-c).$$

Adding this inequality with its similar ones, we obtain

$$16(ab^2+bc^2+ca^2)+4(a+b+c)+(ac^2+ba^2+cb^2)+24abc\ge 20$$

$$\Leftrightarrow 15(ab^2+bc^2+ca^2)+(a+b+c)(ab+bc+ca)+4(a+b+c)+21abc\ge 20$$

$$\Leftrightarrow a+b+c+3(ab^2+bc^2+ca^2)+\frac{21}{5}abc\ge 4.$$

So the inequality (1) is true and the proof is complete.

The maximum value of P is $\frac{16}{21}$ achieved at $(a,b,c)\in\left\{\left(2,\frac{1}{2},0\right),\left(0,2,\frac{1}{2}\right),\left(\frac{1}{2},0,2\right)\right\}$.

1517. If $a,b,c>0$ then:

$$\frac{a^{n+1}b^nc+c^{2n+1}a}{b^{n+1}(a^{n+1}+c^{n+1})}+\frac{b^{n+1}c^na+a^{2n+1}b}{c^{n+1}(a^{n+1}+b^{n+1})}+\frac{c^{n+1}a^nb+b^{2n+1}c}{a^{n+1}(b^{n+1}+c^{n+1})}\ge 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

Walter Janous inequality: a,b,c *and* x,y,z *be positive real numbers then*

$$\frac{x}{y+z}(b+c)+\frac{y}{z+x}(c+a)+\frac{z}{x+y}(a+b)\ge\sqrt{3(ab+bc+ca)}\ (1)$$

$$\frac{a^{n+1}b^nc+c^{2n+1}a}{b^{n+1}(a^{n+1}+c^{n+1})}=\frac{1}{b^{n+1}}\frac{\frac{a^{n+1}b^nc+c^{2n+1}a}{a^{n+1}c^{n+1}}}{\frac{(a^{n+1}+c^{n+1})}{a^{n+1}c^{n+1}}}=\frac{1}{b^{n+1}}\frac{\left(\frac{b^n}{c^n}+\frac{c^n}{a^n}\right)}{\frac{1}{a^{n+1}}+\frac{1}{c^{n+1}}}=$$

$$=\frac{\frac{1}{b^{n+1}}}{\frac{1}{a^{n+1}}+\frac{1}{c^{n+1}}}\left(\frac{b^n}{c^n}+\frac{c^n}{a^n}\right)\ (2)$$

Similarly $\frac{b^{n+1}c^na+a^{2n+1}b}{c^{n+1}(a^{n+1}+b^{n+1})}=\frac{\frac{1}{c^{n+1}}}{\frac{1}{a^{n+1}}+\frac{1}{b^{n+1}}}\left(\frac{c^n}{a^n}+\frac{a^n}{b^n}\right)$ *and*

$$\frac{c^{n+1}a^nb+b^{2n+1}c}{a^{n+1}(b^{n+1}+c^{n+1})}=\frac{\frac{1}{a^{n+1}}}{\frac{1}{b^{n+1}}+\frac{1}{c^{n+1}}}\left(\frac{b^n}{c^n}+\frac{a^n}{b^n}\right)$$

$$\frac{a^{n+1}b^nc+c^{2n+1}a}{b^{n+1}(a^{n+1}+c^{n+1})}+\frac{b^{n+1}c^na+a^{2n+1}b}{c^{n+1}(a^{n+1}+b^{n+1})}+\frac{c^{n+1}a^nb+b^{2n+1}c}{a^{n+1}(b^{n+1}+c^{n+1})}=$$

$$=\frac{\frac{1}{b^{n+1}}}{\frac{1}{a^{n+1}}+\frac{1}{c^{n+1}}}\left(\frac{b^n}{c^n}+\frac{c^n}{a^n}\right)+\frac{\frac{1}{c^{n+1}}}{\frac{1}{a^{n+1}}+\frac{1}{b^{n+1}}}\left(\frac{c^n}{a^n}+\frac{a^n}{b^n}\right)+\frac{\frac{1}{a^{n+1}}}{\frac{1}{b^{n+1}}+\frac{1}{c^{n+1}}}\left(\frac{b^n}{c^n}+\frac{a^n}{b^n}\right)\overset{(1)}{\geq}$$

$$\geq\sqrt{3\left(\left(\frac{b^n}{c^n}\cdot\frac{c^n}{a^n}\right)+\left(\frac{c^n}{a^n}\cdot\frac{a^n}{b^n}\right)+\left(\frac{b^n}{c^n}\cdot\frac{a^n}{b^n}\right)\right)}\overset{AM-GM}{\geq}$$

$$\geq\sqrt{9\left(\left(\left(\frac{b^n}{c^n}\cdot\frac{c^n}{a^n}\right)\cdot\left(\frac{c^n}{a^n}\cdot\frac{a^n}{b^n}\right)\cdot\left(\frac{b^n}{c^n}\cdot\frac{a^n}{b^n}\right)\right)^{\frac{1}{3}}\right)}=3$$

Equality holds for $a=b=c$

1518. **If $a,b,c>0$ then:**

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq 2-\frac{ab+bc+ca}{2(a^2+b^2+c^2)}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=\sum\frac{a}{b+c}=\sum\frac{a^2}{ab+ac}\overset{CBS}{\geq}\frac{(a+b+c)^2}{2(ab+bc+ca)}=$$

$$=\frac{a^2+b^2+c^2+2(ab+bc+ca)}{2(ab+bc+ca)}\overset{\frac{a^2+b^2+c^2}{ab+bc+ca}=x\geq 1}{=}\frac{x}{2}+1$$

We need to show:

$$\frac{x}{2}+1\geq 2-\frac{ab+bc+ca}{2(a^2+b^2+c^2)}\overset{\frac{a^2+b^2+c^2}{ab+bc+ca}=x\geq 1}{=}2-\frac{1}{2x}$$

$$\frac{x+2}{2}\geq\frac{4x-1}{2x}$$

$$x^2-2x+1\geq 0$$

$$(x-1)^2\geq 0\ true$$

Equality holds for $a=b=c$.

1519. Let $a,b,c\geq 0, ab+bc+ca+abc=4$. Prove that :

$$\frac{a}{\sqrt{a^2+8}}+\frac{b}{\sqrt{b^2+8}}+\frac{c}{\sqrt{c^2+8}}\geq 1$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$\frac{a}{\sqrt{a^2+8}}+\frac{b}{\sqrt{b^2+8}}+\frac{c}{\sqrt{c^2+8}}\geq \frac{(a+b+c)^2}{a\sqrt{a^2+8}+b\sqrt{b^2+8}+c\sqrt{c^2+8}}.$$

So it suffices to prove that

$$a\sqrt{a^2+8}+b\sqrt{b^2+8}+c\sqrt{c^2+8}\leq (a+b+c)^2.$$

By AM − GM inequality and the known identity $\frac{a}{a+2}+\frac{b}{a+2}+\frac{c}{a+2}$ $=1$, we have

$$\sum_{cyc} a\sqrt{a^2+8}\leq \sum_{cyc}\frac{a}{2}\left(\frac{a^2+8}{a+2}+a+2\right)=\sum_{cyc} a\left(a+\frac{6}{a+2}\right)=a^2+b^2+c^2+6.$$

So it suffices to prove that 3 $\leq ab+bc+ca$, which is true by AM − GM inequality.

The proof is complete. Equality holds iff $a=b=c=1$.

1520. If $a, \mathrm{b}, \mathrm{c} > 0$ and $abc=1$, then prove that :

$$(a^2+\mathrm{b}^2+\mathrm{c}^2)^3\geq 9(a^3+\mathrm{b}^3+\mathrm{c}^3)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Assigning $\mathrm{b}+\mathrm{c}=x, \mathrm{c}+a=\mathrm{y}, a+\mathrm{b}=\mathrm{z}\Rightarrow x+\mathrm{y}-\mathrm{z}=2\mathrm{c}>0,$
$\mathrm{y}+\mathrm{z}-x=2a>0$ and $\mathrm{z}+x-\mathrm{y}=2\mathrm{b}>0\Rightarrow x+\mathrm{y}>z, \mathrm{y}+\mathrm{z}>x, \mathrm{z}+x>\mathrm{y}$
$\Rightarrow x,\mathrm{y},\mathrm{z}$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $=\mathrm{s},\mathrm{R},\mathrm{r}$ (say); and then : $abc=\mathrm{r}^2\mathrm{s}\rightarrow(1), \sum_{\mathrm{cyc}} a^2=\mathrm{s}^2-8\mathrm{Rr}-2\mathrm{r}^2\rightarrow(2)$ and

$$\sum_{cyc} a^3 = s^3 - 12Rrs \rightarrow (3) \therefore (a^2+b^2+c^2)^3 \geq 9(a^3+b^3+c^3) \overset{abc=1}{\Leftrightarrow}$$

$$\left(\sum_{cyc} a^2\right)^3 \geq 9abc\left(\sum_{cyc} a^3\right) \overset{\text{via (1),(2) and (3)}}{\Leftrightarrow}$$

$$(s^2-8Rr-2r^2)^3 - 9r^2s(s^3-12Rrs) \overset{(*)}{\geq} 0 \text{ and } \because P =$$

$$(s^2-16Rr+5r^2)^3 + (24Rr-30r^2)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0$$

$$\therefore \text{in order to prove } (*), \text{it suffices to prove : LHS of } (*) \overset{?}{\geq} P$$

$$\Leftrightarrow (192R^2-516Rr+237r^2)s^2 \underset{(**)}{\overset{?}{\geq}} r(2560R^3-7296R^2r+4296Rr^2-617r^3)$$

Case 1 $192R^2-516Rr+237r^2 \geq 0$ and then : LHS of $(**) \overset{\text{Gerretsen}}{\geq}$

$$(192R^2-516Rr+237r^2)(16Rr-5r^2) \overset{?}{\geq} \text{RHS of } (**)$$

$$\Leftrightarrow 128t^3-480t^2+519t-142 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)(128t^2-224t+71) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \text{ is true}$$

Case 2 $192R^2-516Rr+237r^2 < 0$ and then : LHS of $(**) \overset{\text{Gerretsen}}{\geq}$

$$(192R^2-516Rr+237r^2)(4R^2+4Rr+3r^2) \overset{?}{\geq} \text{RHS of } (**)$$

$$\Leftrightarrow 192t^4-964t^3+1689t^2-1224t+332 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)\left((t-2)(192t^2-196t+137)+108\right) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (**) \text{ is true} \therefore \text{combining both cases}, (**) \Rightarrow (*) \text{ is true } \forall \Delta ABC$$

$$\therefore (a^2+b^2+c^2)^3 \geq 9(a^3+b^3+c^3)\ \forall a,b,c > 0 \mid abc = 1,$$
$$'' ='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1521. Let $a,b,c \geq 0, ab+bc+ca > 0.$ Prove that :

$$\frac{1}{ab+ac+4bc} + \frac{1}{bc+ba+4ca} + \frac{1}{ca+cb+4ab} \leq \frac{3(a^2+b^2+c^2+ab+bc+ca)}{4(ab+bc+ca)^2}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Multiplying the both sides of the inequality by $4(ab+bc+ca)$, we get the equivalent inequality

$$\frac{a(b+c)}{ab+ac+4bc}+\frac{b(c+a)}{bc+ba+4ca}+\frac{c(a+b)}{ca+cb+4ab}\le\frac{a^2+b^2+c^2}{ab+bc+ca}.$$

We have

$$\blacksquare(ab+bc+ca)\sum_{cyc}\frac{a(b+c)}{ab+ac+4bc}=\sum_{cyc}\left(a(b+c)-\frac{3abc(b+c)}{ab+ac+4bc}\right)$$

$$\overset{AM-GM}{\le}\ 2(ab+bc+ca)-3abc\sum_{cyc}\frac{b+c}{ab+ac+(b+c)^2}$$

$$=2(ab+bc+ca)-\frac{9abc}{a+b+c}\overset{Schur}{\le}\ a^2+b^2+c^2,$$

which completes the proof.

Equality holds iff $a=b=c$ & $a=b, c=0$ and its permutation.

1522. **If $a, b, c>0$ and $a+b+c=3$, then prove that :**

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{a^2+b^2+c^2}\ge\frac{10}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0$, $y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$ $\Rightarrow x, y, z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $=s, R, r$ (say); and then : $\sum_{cyc}a=s\to(1), abc=r^2s\to(2)$,

$$\sum_{cyc}a^2=s^2-8Rr-2r^2\to(3),\sum_{cyc}a^2b^2=r^2((4R+r)^2-2s^2)\to(4)$$

$$\therefore\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{a^2+b^2+c^2}\ge\frac{10}{3}\overset{a+b+c=3}{\Leftrightarrow}$$

$$\left(\sum_{cyc}a\right)^2\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}a^2\right)+\left(\sum_{cyc}a\right)^2.a^2b^2c^2\ge30a^2b^2c^2\left(\sum_{cyc}a^2\right)$$

$$\overset{\text{via }(1),(2),(3)\text{ and }(4)}{\Leftrightarrow}(s^2-8Rr-2r^2)((4R+r)^2-2s^2)+s^2r^2-$$

$$30r^2(s^2-8Rr-2r^2)\overset{(*)}{\geq}0\ and\ \because P=-2s^2(s^2-4R^2-4Rr-3r^2)+$$
$$2(4R^2+8Rr-15r^2)(s^2-16Rr+5r^2)\overset{Gerretsen}{\geq}0\ \therefore \text{in order to prove }(*),$$
$$\text{it suffices to prove : LHS of }(*)\overset{?}{\geq}P\Leftrightarrow 8r^2(15R-13r)(R-2r)\overset{?}{\geq}0$$
$$\rightarrow \text{true via Euler}\Rightarrow(*)\text{ is true}\therefore\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{a^2+b^2+c^2}\geq\frac{10}{3}$$
$$\forall a,b,c>0\,\Big|\,a+b+c=3,''=''\ \text{iff } a=b=c=1\ (QED)$$

1523. Let $a,b,c\geq 0, a+b+c=2.$ Prove that :

$$\frac{bc}{\sqrt{5a+4}}+\frac{ca}{\sqrt{5b+4}}+\frac{ab}{\sqrt{5c+4}}\leq\frac{1}{2}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Morocco

We will first prove the following lemma, that

$$\frac{1}{\sqrt{5a+4}}\leq\frac{3a^2-8a+16}{32}.\quad(1)$$

$$(1)\Leftrightarrow 32\leq(3a^2-8a+16)\sqrt{5a+4}\overset{squaring}{\Leftrightarrow}$$

$$0\leq a(45a^4-204a^3+608a^2-640a+256)$$

$$\Leftrightarrow 0\leq a[(5a^2-16a+13)(3a-2)^2+279a^2-420a+204],$$

which is true since $5a^2-16a+13, 279a^2-420a+204\geq 0$.

Using this lemma, we have

$$\sum_{cyc}\frac{bc}{\sqrt{5a+4}}\leq\sum_{cyc}\frac{3a^2bc-8abc+16bc}{32}=\frac{8(ab+bc+ca)-9abc}{16}=$$

$$=\frac{4(a+b+c)(ab+bc+ca)-9abc}{16}\overset{Schur}{\leq}\frac{(a+b+c)^3}{16}=\frac{1}{2}.$$

The proof is complete. Equality holds iff $a=b=1, c=0$ and its permutation.

1524. Let $a,b,c\ge 0$,

$$a+b+c=ab+bc+ca>0 \text{ then find the minimum value of :}$$

$$P=\frac{3a-b-c}{b^2+c^2}+\frac{3b-c-a}{c^2+a^2}+\frac{3c-a-b}{a^2+b^2}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $p:=a+b+c, q:=ab+bc+ca, r:=abc$. We have $3p=3q\le p^2 \Rightarrow p \ge 3$. Also,

$$P=\sum_{cyc}\frac{4a-p}{b^2+c^2}=\frac{\sum_{cyc}(4a-p)\left[a^2(a^2+b^2+c^2)+b^2c^2\right]}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}=$$

$$=\frac{4\sum a^3.\sum a^2+4abc\sum bc-p(\sum a^2)^2-p\sum b^2c^2}{\sum a^2.\sum b^2c^2-a^2b^2c^2}=$$

$$=\frac{4(p^3-3pq+3r)(p^2-2q)+4qr-p(p^2-2q)^2-p(q^2-2pr)}{(p^2-2q)(q^2-2pr)-r^2}=$$

$$\overset{q=p}{=}\frac{3p^5-16p^4+19p^3+p(14p-20)r}{p^4-2p^3-2p^2(p-2)r-r^2}=f(r).$$

f is clearly incresing, then we have : If $p\ge 4$: $P=f(r)\ge f(0)=\frac{3p^2-16p+19}{p-2}=$

$=\frac{3}{2}+\frac{(p-4)(6p-11)}{2(p-2)}\ge\frac{3}{2}$. If $3\le p\le 4$:

By the fourth degree Schur's inequality, we have

$$r\ge\frac{(p^2-q)(4q-p^2)}{6p}, \text{then}$$

$$P=f(r)\ge f\left(\frac{p(p-1)(4-p)}{6}\right)=\frac{24p^3-36p^2-252p+480}{11p^4-74p^3+171p^2-128p-16}=$$

$$=\frac{3}{2}+\frac{(4-p)(p-3)(33p^2-39p-84)}{2[(p-3)(p(p-3)(11p-8)+24p+16)+32]}\ge\frac{3}{2}.$$

So the maximum value of P is $\frac{3}{2}$ achieved at

$$a=b=c=1\ \&\ a=b=2, c=0 \text{ and permutations.}$$

1525. If $x, y, z \neq k\pi$, $k \in Z$, $\cot^2 x + \cot^2 y + \cot^2 z \leq \frac{3}{7}$ then:

$$\frac{1}{2+\cos 2x}+\frac{1}{2+\cos 2y}+\frac{1}{2+\cos 2z} \geq \frac{12}{5}$$

Proposed by Nguyen Minh Tho-Vietnam

Solution by Tapas Das-India

$$\text{Let } \cot^2 x = a, \cot^2 y = b, \cot^2 z = c \text{ then}$$

$$a+b+c=\cot^2 x+\cot^2 y+\cot^2 z \leq \frac{3}{7}\quad (1)$$

$$2+\cos 2x = 2+\frac{1-\tan^2 x}{1+\tan^2 x} = 2+\frac{\cot^2 x+1}{\cot^2 x-1} = 2+\frac{a-1}{a+1} = \frac{3a+1}{a+1}$$

$$\sum \frac{1}{2+\cos 2x} = \sum \frac{a+1}{3a+1}$$

Lemma :

$$\forall\, t > 0\ \frac{t+1}{3t+1} \geq \frac{47-49t}{50}$$

Proof:

$$\frac{t+1}{3t+1} \geq \frac{47-49t}{50}$$

$$50(t+1) \geq (3t+1)(47-49t)$$

$$50t+50 \geq 92t-147t^2+47$$

$$147t^2-42t+3 \geq 0$$

$$3(7t-1)^2 \geq 0 \ true$$

$$\frac{1}{2+\cos 2x}+\frac{1}{2+\cos 2y}+\frac{1}{2+\cos 2z} = \sum \frac{1}{2+\cos 2x} =$$

$$=\sum\frac{a+1}{3a+1}\overset{lemma}{\geq}\sum\frac{47-49a}{50}=\frac{141-49(a+b+c)}{50}\overset{(1)}{\geq}\frac{141-49\times\frac{3}{7}}{50}=\frac{120}{50}=\frac{12}{5}$$

Equality holds for $a=b=c=\frac{1}{7}$

1526. Let $a,b,c\geq 0, a+b+c+abc=4.$ Prove that :

$$\frac{ab}{4a+4b+c}+\frac{bc}{4b+4c+a}+\frac{ca}{4c+4a+b}\leq\frac{1}{a+b+c}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $p:=a+b+c, q:=ab+bc+ca, r:=abc.$

$$\text{We have } 4=p+r\overset{AM-GM}{\leq}p+\frac{p^3}{27}, \text{then } 3\leq p\leq 4.$$

We have

$$\sum_{cyc}\frac{bc}{4b+4c+a}=\sum_{cyc}\frac{bc}{4p-3a}=\frac{\sum_{cyc}bc(4p-3b)(4p-3c)}{(4p-3a)(4p-3b)(4p-3c)}=$$

$$=\frac{16p^2\sum bc-12p\sum bc(b+c)+9\sum(bc)^2}{64p^3-48p^2\sum a+36p\sum bc-27abc}$$

$$=\frac{16p^2q-12p(pq-3r)+9(q^2-2pr)}{64p^3-48p^3+36pq-27r}=$$

$$\overset{r=4-p}{=}\frac{4p^2q+9q^2-18p^2+72p}{16p^3+36pq+27p-108}\overset{?}{\leq}\frac{1}{p}\Leftrightarrow$$

$$4p(p^2-9)q+9pq^2\leq 34p^3-72p^2+27p-108\ \ (1)$$

By Schur's inequality, we have

$$q\leq\frac{p^3+9r}{4p}=\frac{p^3+9(4-p)}{4p}$$

$$LHS_{(1)}\leq(p^2-9)(p^3-9p+36)+\frac{9(p^3-9p+36)^2}{16p}=$$

$$= 34p^3 - 72p^2 + 27p - 108 - \frac{(p-3)(4-p)(25p^4+175p^3-69p^2-207p+972)}{16p} \le RHS_{(1)},$$

the last inequality is true because $3 \le p \le 4$. So the proof is complete.

Equality holds iff $(a = b = c = 1)$ and $(a = b = 2, c = 0)$ and its permutations.

1527. If $a, b, c > 0$ and $\lambda \le 4$ then :

$$\sum_{cyc} \frac{b+c}{a} \ge \lambda . \frac{\sum_{cyc} a^2}{\sum_{cyc} ab} + (6-\lambda). \frac{\sum_{cyc} ab}{\sum_{cyc} a^2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0$,
$y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y$
$\Rightarrow x, y, z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius
$= s, R, r$ (say); and then : $\sum_{cyc} a = s \to (1), abc = r^2 s \to (2)$,

$$\sum_{cyc} ab = 4Rr + r^2 \to (3) \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (4)$$

$$\therefore \lambda . \frac{\sum_{cyc} a^2}{\sum_{cyc} ab} + (6-\lambda). \frac{\sum_{cyc} ab}{\sum_{cyc} a^2} =$$

$$\frac{\lambda\left(\left(\sum_{cyc} a^2\right)^2 - \left(\sum_{cyc} ab\right)^2\right)}{\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2\right)} + 6.\frac{\sum_{cyc} ab}{\sum_{cyc} a^2} \overset{\lambda \le 4}{\le} \frac{4\left(\left(\sum_{cyc} a^2\right)^2 - \left(\sum_{cyc} ab\right)^2\right)}{\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2\right)} +$$

$$6.\frac{\sum_{cyc} ab}{\sum_{cyc} a^2} \overset{?}{\le} \sum_{cyc} \frac{b+c}{a} = \frac{\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 3abc}{abc} \overset{\text{via } (1),(2),(3) \text{ and } (4)}{\Leftrightarrow}$$

$$\frac{s(4Rr + r^2) - 3r^2 s}{r^2 s} \overset{?}{\ge} \frac{4((s^2 - 8Rr - 2r^2)^2 - (4Rr + r^2)^2) + 6(4Rr + r^2)^2}{(4Rr + r^2)(s^2 - 8Rr - 2r^2)}$$

$$\Leftrightarrow (4R + r)(2R - r)(s^2 - 8Rr - 2r^2) - 2(s^2 - 8Rr - 2r^2)^2 - (4Rr + r^2)^2 \overset{(*)}{\ge} 0$$

and $\because P = -2(s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3) -$

$$r(10R - 11r)(s^2 - 4R^2 - 4Rr - 3r^2) \overset{\text{Doublle-Rouche and Gerretsen}}{\ge} 0, \therefore \text{ in order to prove } (*),$$

it suffices to prove : LHS of $(*) \overset{?}{\ge} P \Leftrightarrow 12t^3 - 22t^2 - 11t + 14 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right)$

$$\Leftrightarrow (t-2)(12t^2 + 2t - 7) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc}\frac{b+c}{a}\geq \lambda.\frac{\sum_{cyc}a^2}{\sum_{cyc}ab}+(6-\lambda).\frac{\sum_{cyc}ab}{\sum_{cyc}a^2}\ \forall\ a,b,c>0 \text{ and } \lambda\leq 4,$$

$$''='' \text{ iff } a=b=c\ (QED)$$

1528. **If $a,b,c>0$ and $\lambda\geq 0$, then :**

$$\sum_{cyc}\frac{a}{(\lambda+1)a^2+\lambda b^2+c^2}\leq \frac{1}{2(\lambda+1)}\sum_{cyc}\frac{1}{a}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a}{(\lambda+1)a^2+\lambda b^2+c^2}\overset{AM-GM}{\leq}\sum_{cyc}\frac{a}{\lambda.2ab+2ac}$$

$$=\frac{1}{2}.\frac{1}{(a\lambda+b)(b\lambda+c)(c\lambda+a)}\sum_{cyc}\left((c\lambda+a)(a\lambda+b)\right)$$

$$\overset{?}{\leq}\frac{1}{2(\lambda+1)}\sum_{cyc}\frac{1}{a}=\frac{1}{2(\lambda+1)abc}.\sum_{cyc}ab$$

$$\Leftrightarrow \lambda\left(\sum_{cyc}a^3b^2-abc\sum_{cyc}ab\right)+\left(\sum_{cyc}a^2b^3-abc\sum_{cyc}ab\right)\underset{(*)}{\overset{?}{\geq}}0$$

$$\text{Now, }\sum_{cyc}a^3b^2=abc\sum_{cyc}\frac{a^2b^2}{bc}\overset{Bergstrom}{\geq}abc\left(\sum_{cyc}ab\right)\text{ and }\because \lambda\geq 0$$

$$\therefore \lambda\left(\sum_{cyc}a^3b^2-abc\sum_{cyc}ab\right)\text{ and also, }\sum_{cyc}a^2b^3=abc\sum_{cyc}\frac{a^2b^2}{ac}\overset{Bergstrom}{\geq}$$

$$abc\left(\sum_{cyc}ab\right)\Rightarrow \sum_{cyc}a^2b^3-abc\sum_{cyc}ab\geq 0\Rightarrow (*)\text{ is true}$$

$$\therefore \sum_{cyc}\frac{a}{(\lambda+1)a^2+\lambda b^2+c^2}\leq \frac{1}{2(\lambda+1)}\sum_{cyc}\frac{1}{a}\ \forall\ a,b,c>0 \text{ and } \lambda\geq 0,$$

$$''='' \text{ iff } a=b=c\ (QED)$$

1529. If $x, y > 0, x + y = xy$ and $n \in \mathbb{N}^*$, then :

$$\sum_{cyc} \frac{1}{nx^2 + (n+2)y^2} \le \frac{1}{4(n+1)}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$x + y = xy \Rightarrow x = y(x-1) \text{ and } \because x > 0 \therefore y(x-1) > 0$$

$$\Rightarrow x > 1 \text{ and similarly, } y > 1$$

$$\text{Now, } \frac{1}{nx^2 + (n+2)y^2} + \frac{1}{ny^2 + (n+2)x^2} \overset{AM-GM}{\le} \frac{1}{n.2xy + 2y^2} + \frac{1}{n.2xy + 2x^2}$$

$$= \frac{1}{2xy} \cdot \frac{x}{nx + y} + \frac{1}{2xy} \cdot \frac{y}{ny + x} = \frac{1}{2xy} \cdot \left(\frac{x}{nx + \frac{x}{x-1}} + \frac{y}{ny + \frac{y}{y-1}} \right)$$

$$= \frac{1}{2xy} \cdot \left(\frac{x-1}{n(x-1)+1} + \frac{y-1}{n(y-1)+1} \right) \overset{Jensen}{\le} \frac{1}{xy} \cdot \frac{\frac{x+y}{2} - 1}{n\left(\frac{x+y}{2} - 1\right) + 1}$$

$$\left(\begin{array}{c} \because f(t) = \dfrac{t-1}{n(t-1)+1} \ \forall t > 1 \text{ and } \forall n \in \mathbb{N}^* \\ \Rightarrow f''(t) = -\dfrac{2n}{(n(t-1)+1)^3} < 0 \Rightarrow f(t) \text{ is concave} \end{array} \right) \overset{x+y=xy}{=} \frac{1}{xy} \cdot \frac{\frac{xy}{2} - 1}{n\left(\frac{xy}{2} - 1\right) + 1}$$

$$= \frac{p-2}{p(np - 2n + 2)} \ (p = xy) \overset{?}{\le} \frac{1}{4(n+1)} \Leftrightarrow np^2 - 6np + 8n - 2p + 8 \overset{?}{\ge} 0$$

$$\Leftrightarrow n(p-2)(p-4) - 2(p-4) \overset{?}{\ge} 0 \Leftrightarrow (p-4)(n(p-2) - 2) \underbrace{\overset{?}{\ge}}_{(*)} 0$$

$$\text{Now, } p = xy = x + y \overset{AM-GM}{\ge} 2\sqrt{p} \Rightarrow \sqrt{p} \ge 2 \Rightarrow p \ge 4 \text{ and } p - 2 \ge 2 \therefore \text{LHS of } (*)$$

$$\ge (p-4)(2n-2) = 2(p-4)(n-1) \ge 0 \because p \ge 4 \text{ and } n \ge 1 \ (\because n \in \mathbb{N}^*)$$

$$\Rightarrow (*) \text{ is true } \therefore \sum_{cyc} \frac{1}{nx^2 + (n+2)y^2} \le \frac{1}{4(n+1)} \ \forall x, y > 0 \mid x + y = xy \text{ and}$$

$$\forall n \in \mathbb{N}^*, " = " \text{ iff } x = y = 2 \ (QED)$$

1530. If $x, y, z > 0$ and $xy + yz + zx + xyz = 4$ then prove that :

$$\sqrt{x} + \sqrt{y} + \sqrt{z} > 2\sqrt{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \left((2+y)(2+z) \right) - (2+x)(2+y)(2+z) = 4 - (xy + yz + zx + xyz)$$

$$= 0 \therefore \sum_{cyc} \frac{1}{2+x} = 1 \to (m) \text{ and } \because \frac{1}{2+x} \overset{x>0}{<} \frac{1}{2} \therefore \text{we can set}: \frac{1}{2+x} =$$

$$\frac{1}{2} - a \left(a > 0 \text{ and } a < \frac{1}{2}\right) \therefore x + 2 = \frac{2}{1-2a} \Rightarrow x = \frac{2}{1-2a} - 2 = \frac{2a}{\frac{1}{2} - a} \to (1)$$

$$\text{Similarly, we set}: \frac{1}{2+y} = \frac{1}{2} - b \text{ and } \frac{1}{2+z} = \frac{1}{2} - c \therefore 1 \overset{\text{via }(m)}{=}$$

$$\frac{1}{2+x} + \frac{1}{2+y} + \frac{1}{2+z} = \frac{1}{2} - a + \frac{1}{2} - b + \frac{1}{2} - c \Rightarrow a + b + c = \frac{1}{2} \to (i)$$

$$\therefore (1) \text{ and } (i) \Rightarrow x = \frac{2a}{b+c} \text{ and analogously}, y = \frac{2b}{c+a} \text{ and } z = \frac{2c}{a+b}$$

$$\text{and hence}: \sqrt{x} + \sqrt{y} + \sqrt{z} > 2\sqrt{2} \Leftrightarrow \sum_{cyc} \sqrt{\frac{a}{b+c}} \overset{(*)}{>} 2$$

Assigning $b + c = A, c + a = B, a + b = C \Rightarrow A + B - C = 2c > 0$,
$B + C - A = 2a > 0$ and $C + A - B = 2b > 0 \Rightarrow A + B > C, B + C > A, C + A > B$
$\Rightarrow A, B, C$ form sides of a triangle with semiperimeter, circumradius and inradius

$$= s, R, r \text{ (say) yielding } 2\sum_{cyc} a = \sum_{cyc} A = 2s \Rightarrow \sum_{cyc} a = s$$

$$\Rightarrow a = s - A, b = s - B, c = s - C \therefore (\text{LHS of } (*))^2 = \left(\sum_{cyc} \sqrt{\frac{s-A}{A}}\right)^2$$

$$= \sum_{cyc} \frac{s-A}{A} + 2\sqrt{\frac{(s-B)(s-C)}{BC}} = \frac{s(s^2 + 4Rr + r^2)}{4Rrs} - 3 +$$

$$2\sum_{cyc} \frac{\sin\frac{\alpha}{2} \cos\frac{\beta-\gamma}{2}}{\cos\frac{\beta-\gamma}{2}} \; (\alpha, \beta, \gamma \to \text{angles of triangle with sides A, B and C})$$

$$\geq \frac{s^2 + 4Rr + r^2}{4Rr} - 3 + \sum_{cyc} \left(2\cos\frac{\beta+\gamma}{2} \cos\frac{\beta-\gamma}{2}\right)$$

$$\left(\because 0 < \cos\frac{\beta-\gamma}{2} \leq 1 \text{ and analogs}\right) = \frac{s^2 + 4Rr + r^2}{4Rr} - 3 + \sum_{cyc} (\cos\beta + \cos\gamma)$$

$$= \frac{s^2 + 4Rr + r^2}{4Rr} - 3 + \frac{2(R+r)}{R} \overset{?}{>} 4 \Leftrightarrow \frac{s^2 + 12Rr + 9r^2}{4Rr} \overset{?}{>} 7$$

$$\Leftrightarrow s^2 - 16Rr + 5r^2 + 4r^2 \overset{?}{>} 0 \to \text{true} \because s^2 - 16Rr + 5r^2 + 4r^2 \overset{\text{Gerretsen}}{\geq} 4r^2 > 0$$

$$\Rightarrow (*) \text{ is true} \therefore \sqrt{x} + \sqrt{y} + \sqrt{z} > 2\sqrt{2} \; \forall \, x, y, z > 0 \,|\, xy + yz + zx + xyz = 4 \text{ (QED)}$$

1531. If $x, y, z > 0$ and $\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1$ then:

$$xyz \geq 8$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1$$

$$\frac{(y+1)(z+1) + (x+1)(z+1) + (x+1)(y+1)}{(x+1)(y+1)(z+1)} = 1$$

$$\frac{3 + 2\sum x + \sum yz}{1 + \sum x + \sum xy + xyz} = 1$$

$$3 + 2\sum x + \sum yz = 1 + \sum x + \sum xy + xyz$$

$$-2 + xyz = \sum x$$

$$-2 + xyz \overset{AM-GM}{\geq} 3\sqrt[3]{xyz} \text{ or } -2 + a^3 \overset{xyz=a^3>0}{\geq} 3a$$

$$a^3 - 3a - 2 \geq 0 \text{ or } (a-2)(a+1)^2 \geq 0$$

$$as\ (a+1)^2 > 0 \text{ so satisfy the inequality } (a-2) \geq 0$$
$$a \geq 2$$

$$\sqrt[3]{xyz} \geq 2 \text{ or } xyz \geq 8$$

Equality holds for $x = y = z = 2$.

1532. If $a, b, c > 0, a + b + c = 3$ then:

$$ab + bc + ca \leq \frac{3abc + 9\sqrt[3]{a^2b^2c^2} - 27\sqrt[3]{abc} + 27}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\text{Since } a, b, c > 0 \text{ so } abc > 0 \text{ and } abc \overset{AM-GM}{\leq} \left(\frac{a+b+c}{3}\right)^3 \overset{a+b+c=3}{=} 1, 0 < abc \leq 1$$

$$\frac{(a+b+c)^2}{3}\geq ab+bc+ca$$

$$ab+bc+ca\leq 3\ (as\ \ a+b+c=3)(1)$$

$$ab+bc+ca\leq\frac{3abc+9\sqrt[3]{a^2b^2c^2}-27\sqrt[3]{abc}+27}{4}$$

$$3\overset{(1)}{\leq}\frac{3abc+9\sqrt[3]{a^2b^2c^2}-27\sqrt[3]{abc}+27}{4}$$

$$12\overset{abc=x^3}{\leq}3x^3+9x^2-27x+27$$

$$x^3+3x^2-9x+5\geq 0$$

$$(x-1)^2(x+5)\geq 0$$

$$true\ as\ 0<x^3=abc\leq 1$$

Equality holds for $a=b=c=1$.

1533. If $a,b>0$, $\frac{4}{a^2}+\frac{1}{b^2}=\frac{5}{4}$ then:

$$4a+b\geq 10$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{5}{4}=\frac{4}{a^2}+\frac{1}{b^2}=\frac{64}{16a^2}+\frac{1}{b^2}=\frac{4^3}{(4a)^2}+\frac{(1)^3}{b^2}\overset{RADON}{\geq}\frac{(4+1)^3}{(4a+b)^2}=\frac{5^3}{(4a+b)^2}$$

$$\frac{5}{4}\geq\frac{5^3}{(4a+b)^2}$$

$$(4a+b)^2\geq 100$$

$$4a+b\geq 10$$

Equality holds for $a=b=2$.

1534. If $a,b,c>0$, $ab+bc+ca=3$ then:

$$\sqrt{\frac{bc}{a^4+7}}+\sqrt{\frac{ca}{b^4+7}}+\sqrt{\frac{ab}{c^4+7}}\leq\frac{3\sqrt{2}}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$a^4+7=(a^4+1)+6\overset{AM-GM}{\geq}2a^2+6=2(a^2+3)=$$

$$\overset{ab+bc+ca=3}{=}2(a^2+ab+bc+ca)=2(a+b)(a+c)\quad (1)$$

$$\sqrt{\frac{bc}{a^4+7}}\overset{(1)}{\leq}\sqrt{\frac{bc}{2(a+b)(a+c)}}=\frac{1}{\sqrt{2}}\sqrt{\frac{b}{a+b}\cdot\frac{c}{a+c}}\overset{AM-GM}{\leq}\frac{1}{2\sqrt{2}}\left(\frac{b}{a+b}+\frac{c}{a+c}\right)$$

$$Similary:\sqrt{\frac{ca}{b^4+7}}\leq\frac{1}{2\sqrt{2}}\left(\frac{c}{b+c}+\frac{a}{b+a}\right)\ and$$

$$\sqrt{\frac{ab}{c^4+7}}\leq\frac{1}{2\sqrt{2}}\left(\frac{a}{c+a}+\frac{b}{c+b}\right)$$

$$Using\ above\ result\ we\ get:$$

$$\sqrt{\frac{bc}{a^4+7}}+\sqrt{\frac{ca}{b^4+7}}+\sqrt{\frac{ab}{c^4+7}}\leq$$

$$\leq\frac{1}{2\sqrt{2}}\left(\left(\frac{b}{a+b}+\frac{c}{a+c}\right)+\left(\frac{c}{b+c}+\frac{a}{b+a}\right)+\left(\frac{a}{c+a}+\frac{b}{c+b}\right)\right)=$$

$$\overset{grouping\ the\ terms}{=}\frac{1}{2\sqrt{2}}\left(\left(\frac{b}{a+b}+\frac{a}{a+b}\right)+\left(\frac{c}{a+c}+\frac{a}{a+c}\right)+\left(\frac{b}{c+b}+\frac{c}{c+b}\right)\right)=$$

$$=\frac{1}{2\sqrt{2}}\left(\frac{a+b}{a+b}+\frac{a+c}{a+c}+\frac{b+c}{b+c}\right)=\frac{3}{2\sqrt{2}}=\frac{3\sqrt{2}}{4}$$

Equality holds for $a=b=c=1$.

1535. **If $a,b,c>0$ then:**

$$\frac{\frac{a^7}{c^2}+\frac{b^6}{c}}{a^4+b^4}+\frac{\frac{b^7}{a^2}+\frac{c^6}{a}}{b^4+c^4}+\frac{\frac{c^7}{b^2}+\frac{a^6}{b}}{c^4+a^4}\geq 3\sqrt[3]{abc}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$Walter\ Janous\ inequaliy: x,y,z;\ A,B,C>0\ then:$$

$$\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)\ge\sqrt{3(AB+BC+CA)}\ (1)$$

$$\frac{\frac{a^7}{c^2}+\frac{b^6}{c}}{a^4+b^4}=\frac{\frac{1}{c^4}(a^7c^2+b^6c^3)}{a^4+b^4}=\frac{\frac{1}{c^4}\frac{(a^7c^2+b^6c^3)}{a^4b^4}}{\frac{a^4+b^4}{a^4b^4}}=$$

$$=\frac{\left(\frac{1}{c^4}\right)}{\frac{1}{a^4}+\frac{1}{b^4}}\left(\frac{a^3c^2}{b^4}+\frac{b^2c^3}{a^4}\right)=\frac{z}{x+y}(A+B)\ (2)$$

$$\left(x=\frac{1}{a^4},y=\frac{1}{b^4}z=\frac{1}{c^4};\ A=\frac{b^2c^3}{a^4},B=\frac{a^3c^2}{b^4},C=\frac{b^3a^2}{c^4}\right)$$

$$\frac{\frac{a^7}{c^2}+\frac{b^6}{c}}{a^4+b^4}+\frac{\frac{b^7}{a^2}+\frac{c^6}{a}}{b^4+c^4}+\frac{\frac{c^7}{b^2}+\frac{a^6}{b}}{c^4+a^4}=\sum\frac{\frac{a^7}{c^2}+\frac{b^6}{c}}{a^4+b^4}\overset{(2)}{=}\sum\frac{y}{x+z}(C+A)\overset{(1)}{\ge}$$

$$\ge\sqrt{3(AB+BC+CA)}\overset{AM-GM}{\ge}\sqrt{3\times3\sqrt[3]{A^2B^2C^2}}=3\sqrt[3]{ABC}=$$

$$=3\sqrt[3]{\left(\frac{b^2c^3}{a^4}\cdot\frac{a^3c^2}{b^4}\cdot\frac{b^3a^2}{c^4}\right)}=3\sqrt[3]{abc}$$

Equality holds for a=b=c=1.

1536. For $a,b,c\ge0$ prove that :

$$\sum_{cyc}\frac{1}{a^3+3abc+b^3}\ge\frac{18}{5\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

When exactly one variable $=0$; WLOG we may assume $a=0$, and then :

$$LHS\ge RHS\Leftrightarrow\frac{1}{b^3}+\frac{1}{c^3}\ge\frac{18}{5(b^2c+bc^2)}\Leftrightarrow(b+c)^2(b^2-bc+c^2)\ge\frac{18}{5}.b^2c^2$$

$$\to\text{true}\because(b+c)^2(b^2-bc+c^2)\overset{AM-GM}{\ge}4b^2c^2>\frac{18}{5}.b^2c^2\text{ and now we}$$

focus on the scenario when : $a,b,c>0$ and assigning $b+c=x, c+a=y$, $a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow$ x,y,z form sides of a triangle XYZ with semiperimeter, circumradius & inradius $=s,R,r$ (say); $\sum_{cyc}a=s, abc=r^2s, \sum_{cyc}ab=4Rr+r^2, \sum_{cyc}a^3=s^3-12Rrs,$

$$\sum_{cyc} a^3b^3 = (4Rr+r^2)^3 - 12Rr^3s^2 \text{ and with } m = s^3 - 12Rrs, \text{LHS} =$$

$$\frac{\left(\sum_{cyc} a^3\right)^2 + \sum_{cyc} a^3b^3 + 12abc\sum_{cyc} a^3 + 27a^2b^2c^2}{3abc\left(\sum_{cyc} a^3\right)^2 + \left(\sum_{cyc} a^3\right)\left(\sum_{cyc} a^3b^3\right) + 18a^2b^2c^2\sum_{cyc} a^3 + 3abc\sum_{cyc} a^3b^3 + 26a^3b^3c^3}$$

$$= \frac{m^2 + (4Rr+r^2)^3 - 12Rr^3s^2 + 12r^2sm + 27r^4s^2}{(3r^2sm + 18r^4s^2)m + (m+3r^2s)((4Rr+r^2)^3 - 12Rr^3s^2) + 26r^6s^3} \overset{?}{\geq} \text{RHS}$$

$$\frac{18}{5(s(4Rr+r^2) - 3r^2s)} \Leftrightarrow (5R-16r)s^6 - r(120R^2 - 498Rr + 111r^2)s^4 +$$

$$r^2(432R^3 - 3948R^2r + 1605Rr^2 - 189r^3)s^2 +$$

$$r^3(3776R^4 + 1808R^3r - 60R^2r^2 - 133Rr^3 - 16r^4) \boxed{\overset{?}{\underset{(*)}{\geq}}} 0 \text{ and } \because P =$$

$$(5R-10r)(s^2 - 16Rr + 5r^2)^3 - 6rs^2(s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R+r)^3) +$$

$$r(96R^2 - 177Rr + 51r^2)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen and Double-Rouche}}{\geq} 0 \therefore \text{to prove } (*),$$

$$\text{it suffices to prove : LHS of } (*) \overset{?}{\geq} P \Leftrightarrow (48R^3 - 204R^2r - 96Rr^2 + 57r^3)s^2$$

$$\overset{?}{\underset{(**)}{\geq}} r(320R^4 - 2320R^3r - 564R^2r^2 + 173Rr^3 + 41r^4) \because r^3s^2 \overset{\text{Gerretsen}}{\geq} 16Rr^4 - 5r^5$$

$$\therefore \text{LHS of } (**) \geq 4(12R^3 - 51R^2r - 24Rr^2 + 14r^3) + 16Rr^4 - 5r^5 \overset{?}{\geq} \text{RHS of } (**)$$

$$\Leftrightarrow 4(12R^3 - 51R^2r - 24Rr^2 + 14r^3)s^2 + 3r^4(R-2r)$$

$$\overset{?}{\geq} 4r(80R^4 - 580R^3r - 141R^2r^2 + 40Rr^3 + 10r^4) \text{ and } \because 3r^4(R-2r) \overset{\text{Euler}}{\geq} 0$$

$$\therefore \text{to prove } (**), \text{it suffices to prove : } (12R^3 - 51R^2r - 24Rr^2 + 14r^3)s^2$$

$$\boxed{\overset{?}{\underset{(1)}{\geq}}} r(80R^4 - 580R^3r - 141R^2r^2 + 40Rr^3 + 10r^4)$$

Case 1 $12R^3 - 51R^2r - 24Rr^2 + 14r^3 \geq 0$ *and then* : LHS of (1) $\overset{\text{Gerretsen}}{\geq}$

$$(12R^3 - 51R^2r - 24Rr^2 + 14r^3)(16Rr - 5r^2) \overset{?}{\geq} \text{RHS of } (1)$$

$$\Leftrightarrow 28t^4 - 74t^3 + 3t^2 + 76t - 20 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow$$

$$(t-2)\left((t-2)(28t^2 + 38t + 43) + 96\right) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (1) \text{ is true}$$

Case 2 $12R^3 - 51R^2r - 24Rr^2 + 14r^3 < 0$ *and then* : LHS of (1) $\overset{\text{Rouche}}{\geq}$

$$(12R^3 - 51R^2r - 24Rr^2 + 14r^3)\left(2R^2 + 10Rr - r^2 + 2(R-2r).\sqrt{R^2 - 2Rr}\right) \overset{?}{\geq}$$

$$\text{RHS of } (1) \Leftrightarrow 2(R-2r)(12R^4 - 7R^3r - 9R^2r^2 - 28Rr^3 + 6r^4) \overset{?}{\geq}$$

$$2(R-2r).\sqrt{R^2 - 2Rr}.\left(-(12R^3 - 51R^2r - 24Rr^2 + 14r^3)\right) \Leftrightarrow$$

$$(12R^4 - 7R^3r - 9R^2r^2 - 28Rr^3 + 6r^4)^2 \overset{?}{\geq}$$

$$(R^2 - 2Rr)(12R^3 - 51R^2r - 24Rr^2 + 14r^3)^2$$

$$\Leftrightarrow 1344t^7-4640t^6+720t^5+7037t^4-612t^3-864t^2+56t+36\overset{?}{\geq}0 \text{ and}$$

$$\because t^4\overset{\text{Euler}}{\geq}16 \therefore \text{it suffices to prove}:$$

$$336t^7-1160t^6+180t^5+1759t^4-153t^3-216t^2+14t+13\underset{(2)}{\overset{?}{\geq}}0 \text{ and} \because Q=$$

$$336(t-2)^7+3544(t-2)^6+14484(t-2)^5+28039(t-2)^4\overset{\text{Euler}}{\geq}0$$

$$\therefore \text{in order to prove }(2),\text{it suffices to prove : LHS of }(2)\overset{?}{\geq}Q$$

$$\Leftrightarrow 23679t^3-139200t^2+268462t-168931\underset{(\bullet)}{\overset{?}{\geq}}0\Leftrightarrow$$

$$\frac{1}{125}\left((118395t-175062)(5t-11)^2+66127-24865\right)\overset{?}{\geq}0\to\text{true}$$

$$\forall t\in\left[2,\frac{66127}{24865}\right) \text{ and when}: t\geq\frac{66127}{24865},\text{LHS of }(\bullet)=$$

$$\frac{1}{32}\left((47358t-65289)(4t-9)^2+53978t-117383\right)>0\Rightarrow(2)\Rightarrow(1)\text{ is true}$$

$$\therefore \text{combining all cases},(1)\Rightarrow(**)\Rightarrow(*)\text{ is true }\forall\Delta XYZ\therefore\sum_{cyc}\frac{1}{a^3+3abc+b^3}$$

$$\geq\frac{18}{5\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)}\ \forall a,b,c\geq0,''=''\text{ iff }a=b=c>0\text{ (QED)}$$

1537. **If $x,y,z>0\,,xy+yz+zx=1$ then:**

$$\sum\frac{x^n\sqrt{x}}{yz}\geq\frac{3}{\left(\sqrt{3}\right)^n}\sum x$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{xy+yz+zx}{3}\overset{AM-GM}{\geq}(xyz)^{\frac{2}{3}}\text{ or }(xyz)^{\frac{2}{3}}\overset{xy+yz+zx=1}{\leq}\frac{1}{3}\text{ or }xyz\leq\frac{1}{3\sqrt{3}}\quad(1)$$

$$\sum x=\sqrt{\left(\sum x\right)^2}\geq\sqrt{3(xy+yz+zx)}\overset{xy+yz+zx=1}{=}\sqrt{3}\quad(2)$$

$$\sum\frac{x^n\sqrt{x}}{yz}=\sum\frac{x^{n+1}\sqrt{x}}{xyz}\overset{(1)}{\geq}\sum\frac{x^{n+1}\sqrt{x}}{\frac{1}{3\sqrt{3}}}=3\sqrt{3}\sum x^{n+1}\sqrt{x}\overset{Chebyshev}{\geq}$$

$$\geq 3\sqrt{3}\cdot\frac{1}{3}\left(\sum x^{n+1}\right)\left(\sum\sqrt{x}\right)\overset{CBS}{\geq}\sqrt{3}\frac{1}{3^n}\left(\sum x\right)^{n+1}\left(\sum\sqrt{x}\right)\overset{(2)}{\geq}$$

$$\geq\sqrt{3}\frac{1}{3^n}\left(\sqrt{3}\right)^{n+1}\left(\sum\sqrt{x}\right)=\sqrt{3}.\frac{(\sqrt{3})(\sqrt{3})^n}{(\sqrt{3})^{2n}}\left(\sum\sqrt{x}\right)=\frac{3}{(\sqrt{3})^n}\sum x$$

$$Equality\ holds\ for\ x=y=z=\frac{1}{\sqrt{3}}$$

1538. Let $a,b,c\geq 0, a+b+c=3.$ Prove that :

$$\sqrt{6a+6b+(a-b)^2}+\sqrt{6b+6c+(b-c)^2}+\sqrt{6c+6a+(c-a)^2}\geq 6\sqrt{3}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x:=b+c, y:=c+a, z:=a+b.$ We have $x+y+z=6.$

The desired inequality is equivalent to

$$\sqrt{6x+(y-z)^2}+\sqrt{6y+(z-x)^2}+\sqrt{6z+(x-y)^2}\geq 6\sqrt{3}$$

$$\overset{squaring}{\Leftrightarrow}\sum_{cyc}\sqrt{[6y+(z-x)^2][6z+(x-y)^2]}\geq 3(xy+yz+zx).$$

By CBS inequality, we have

$$\sum_{cyc}\sqrt{[6y+(z-x)^2][6z+(x-y)^2]}\geq\sum_{cyc}\left[6\sqrt{yz}+(x-y)(x-z)\right]\overset{?}{\geq}3(xy+yz+zx)$$

$$\Leftrightarrow (x+y+z)\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right)+x^2+y^2+z^2\geq 4(xy+yz+zx)$$

which is true by summing the two following true inequalities.

$$(x+y)\sqrt{xy}+(y+z)\sqrt{yz}+(z+x)\sqrt{zx}\geq 2xy+2yz+2zx.$$

$$x^2+y^2+z^2+\sqrt{xyz}\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\geq x^2+y^2+z^2+\frac{9xyz}{x+y+z}\geq 2(xy+yz+zx)$$

The proof is complete. Equality holds iff

$a=b=c=1$ & $a=b=0, c=3$ and its permutation.

1539. Let $a,b,c\ge 0, ab+bc+ca>0$. Prove that :

$$\frac{7a^2+6ab+7b^2}{a^2+b^2}+\frac{7b^2+6bc+7c^2}{b^2+c^2}+\frac{7c^2+6ca+7a^2}{c^2+a^2}\ge\frac{90(ab+bc+ca)}{(a+b+c)^2}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

The desired inequality can be rewritten as follows

$$\frac{(a+b)^2}{a^2+b^2}+\frac{(b+c)^2}{b^2+c^2}+\frac{(c+a)^2}{c^2+a^2}+4\ge\frac{30(ab+bc+ca)}{(a+b+c)^2}.$$

WLOG, we assume that $a+b+c=1$.

$$\text{Let } p:=a+b+c, q:=ab+bc+ca\le\frac{p^2}{3}=\frac{1}{3}, r:=abc.$$

By CBS inequality, we have

$$\sum_{cyc}\frac{(b+c)^2}{b^2+c^2}\cdot\sum_{cyc}(b^2+c^2)(5a+4)^2\ge\left(\sum_{cyc}(b+c)(5a+4)\right)^2=4(4+5q)^2,$$

So it suffices to prove that

$$\frac{4(4+5q)^2}{50\sum_{cyc}(bc)^2+40\sum_{cyc}a(b^2+c^2)+32\sum_{cyc}a^2}+4\ge 30q$$

$$\Leftrightarrow\frac{(4+5q)^2}{25(q^2-2pr)+20(pq-3r)+16(p^2-2q)}\ge 15q-2,$$

The last inequality is true for $15q-2\le 0$. Otherwise, it is equivalent to

$$48-224q+255q^2-375q^3+110(15q-2)r\ge 0.$$

From the known identity

$$0\le(a-b)^2(b-c)^2(c-a)^2=-27r^2+2(9pq-2p^3)r+p^2q^2-4q^3$$

It follows

$$r\ge\frac{-2p^3+9pq-2\sqrt{(p^2-3q)^3}}{27}=\frac{-2+9q-2\sqrt{(1-3q)^3}}{27}.$$

So it suffices to prove that

$$48-224q+255q^2-375q^3+\frac{110(15q-2)\left(-2+9q-2\sqrt{(1-3q)^3}\right)}{27}\ge 0$$

$$\Leftrightarrow(1-3q)(1736-6120q+3375q^2)\ge 220(15q-2)\sqrt{(1-3q)^3}$$

$$\overset{Squaring}{\Leftrightarrow}\quad 27(1-3q)^2(8-25q)^2(1632-80q+675q^2)\ge 0,$$

which is true and the proof is complete.

Equality holds iff $\left(q=\frac{1}{3}\Leftrightarrow a=b=c\right)$ and

$\left(q = \frac{8}{25} \Leftrightarrow a = b = 2c\right)$ and its permutations.

1540. Let $a, b, c \geq 0, ab + bc + ca > 0$. Prove that:

$$\sqrt{a^2 + 14ab + b^2} + \sqrt{b^2 + 14bc + c^2} + \sqrt{c^2 + 14ca + a^2} \geq 12 \cdot \frac{ab + bc + ca}{a + b + c}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Firstly, we have

$$\sqrt{b^2 + 14bc + c^2} - (b + c) = \frac{12bc}{\sqrt{4(b + c)^2 - 3(b - c)^2} + (b + c)} \geq \frac{12bc}{2(b + c) + (b + c)}$$
$$= \frac{4bc}{b + c},$$
$$\Rightarrow \sqrt{b^2 + 14bc + c^2} \geq b + c + \frac{4bc}{b + c} \quad \text{(and analogs)},$$

with equality for $b = c > 0$ or $bc = 0$.
Using this result, it suffices to prove that

$$a + b + c + \frac{2ab}{a + b} + \frac{2bc}{b + c} + \frac{2ca}{c + a} \geq \frac{6(ab + bc + ca)}{a + b + c}.$$

Multiplying the last inequality by $a + b + c$, we get the equivalent form

$$a^2 + b^2 + c^2 + 2abc\left(\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a}\right) \geq 2(ab + bc + ca). \quad (1)$$

By CBS and Schur inequalities, we have

$$LHS_{(1)} \geq a^2 + b^2 + c^2 + 2abc.\frac{9}{2(a + b + c)} \geq 2(ab + bc + ca).$$

So the proof is complete. Equality holds iff $a = b = c$ and
$a = b > 0, c = 0$ and its permutations.

1541. Let $a, b, c \geq 0, ab + bc + ca + abc = 4$. Prove that :

$$\frac{a + bc}{b + c} + \frac{b + ca}{c + a} + \frac{c + ab}{a + b} \geq 3$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

The desired inequality is equivalent to

$$(a + b + c + ab + bc + ca)\left(\frac{1}{b + c} + \frac{1}{c + a} + \frac{1}{a + b}\right) \geq 6 + a + b + c.$$

Let $p := a+b+c, q := ab+bc+ca, r := abc$. We have $4 = q+r \leq \frac{p^2}{3}+\frac{p^3}{27}$, then $p \geq 3$.

By CBS inequality, we have

$$\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b} \geq \frac{\left[\sum_{cyc}(a+2)\right]^2}{\sum_{cyc}(b+c)(a+2)^2} = \frac{(p+6)^2}{(pq-3r)+8q+8p} \overset{?}{\geq} \frac{p+6}{p+q}$$

$$\Leftrightarrow p^2-2p+12 \geq 5q.$$

▪If $p \geq 4$: We have $5q \leq 20 = p^2-2p+12-(p-4)(p+2) \leq p^2-2p+12$.

▪If $3 \leq p \leq 4$: We have $4pq \overset{Schur}{\leq} p^3+9r = p^3+9(4-q)$, then

$$5q \leq \frac{5(p^3+36)}{4p+9} = p^2-2p+12-\frac{(p-3)(4-p)(p+6)}{4p+9} \leq p^2-2p+12.$$

Equality holds if $a=b=c=1$ & $a=b=2, c=0$ and permutations.

1542. Let $a,b,c \geq 0, a+b+c=3$. Prove that :

$$\frac{1}{a^2b+2}+\frac{1}{b^2c+2}+\frac{1}{c^2a+2} \geq \frac{7-abc}{6}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $p := a+b+c = 3,\ \ q := ab+bc+ca,\ \ r := abc \leq \frac{p^3}{27} = 1$.

The desired inequality is equivalent to

$$16+8r-7r^3+r^4 \geq 4(1-r)(a^2b+b^2c+c^2a)+2r(4-r)(ab^2+bc^2+ca^2).\quad (1)$$

WLOG, we assume that $b = mid(a,b,c)$. By AM − GM inequality, we have

$$ab^2+bc^2+ca^2 = b(c+a)^2-abc+a(b-c)(b-a)$$
$$\leq \frac{1}{2}\left(\frac{2b+(c+a)+(c+a)}{3}\right)^3-abc+0$$

then $ab^2+bc^2+ca^2 \leq 4-r$. Similarly, we get $a^2b+b^2c+c^2a \leq 4-r$.

▪If $2-6r+r^2 \geq 0$, we have

$$RHS_{(1)}=4(1-r)(a^2b+b^2c+c^2a)+2r(4-r)[pq-3r-(a^2b+b^2c+c^2a)]$$

$$=2(2-6r+r^2)(a^2b+b^2c+c^2a)+6r(4-r)q-6r(4-r)$$

$$\overset{Schur}{\leq}2(2-6r+r^2)(4-r)+6r(4-r).\frac{p^3+9r}{4p}-6r(4-r)$$

$$\overset{p=3}{=}\frac{32-36r+61r^2-13r^3}{2}=16+8r-7r^3+r^4-\frac{r(1-r)(60-r-2r^2)}{2}$$

$\overset{r\leq 1}{\leq}16+8r-7r^3+r^4=LHS_{(1)}$. If $2-6r+r^2\leq 0$, we have

$$RHS_{(1)}=4(1-r)[pq-3r-(ab^2+bc^2+ca^2)]+2r(4-r)(ab^2+bc^2+ca^2)$$

$$=(1-r).4pq-2(2-6r+r^2)(a^2b+b^2c+c^2a)-12r(1-r)$$

$$\overset{Schur}{\leq}(1-r)(p^3+9r)-2(2-6r+r^2)(4-r)-12r(1-r)$$

$$\overset{p=3}{=}11+22r-17r^2+2r^3$$

$$=16+8r-7r^3+r^4-(1-r)[(1-r)(5-4r+4r^2)+3r^3]$$

$$\overset{r\leq 1}{\leq}16+8r-7r^3+r^4=LHS_{(1)}.$$

Equality holds iff $(a=b=c=1)$ and $(a=2,b=1,c=0)$ and its cyclic permutations.

1543. Let $a,b,c\geq 0, a+b+c=1$. Prove that :

$$a\sqrt{a^2+4b}+b\sqrt{b^2+4c}+c\sqrt{c^2+4a}\geq 1$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$a\sqrt{a^2+4b}+b\sqrt{b^2+4c}+c\sqrt{c^2+4a}\geq$$

$$\geq a\sqrt{a^2+4b(a+b)}+b\sqrt{b^2+4c(b+c)}+c\sqrt{c^2+4a(c+a)}$$

$$=a(a+2b)+b(b+2c)+c(c+2a)=(a+b+c)^2=1.$$

Equality holds iff $abc=0$ and $a+b+c=1$.

1544. **Let a,b,c be real numbers with $a+b+c=3$. Prove that :**

$$\frac{a-1}{b^2+bc+c^2}+\frac{b-1}{c^2+ca+a^2}+\frac{c-1}{a^2+ab+b^2}\geq 0$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$3\sum_{cyc}\frac{a-1}{b^2+bc+c^2}=\sum_{cyc}\frac{2a-b-c}{b^2+bc+c^2}=$$

$$=\sum_{cyc}\left(\frac{a-b}{b^2+bc+c^2}-\frac{c-a}{b^2+bc+c^2}\right)=$$

$$=\sum_{cyc}\left(\frac{b-c}{c^2+ca+a^2}-\frac{b-c}{a^2+ab+b^2}\right)=\sum_{cyc}\frac{(a+b+c)(b-c)^2}{(a^2+ab+b^2)(c^2+ca+a^2)}\geq 0$$

Equaliy holds iff $a=b=c=1$.

1545. **Given real numbers a,b,c satisfying $a+b+c=3$. Prove that :**

$$\sqrt{a^2+b^2+3}+\sqrt{b^2+c^2+3}+\sqrt{c^2+a^2+3}\geq\sqrt{5(a^2+b^2+c^2+6)}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We assume that $a=max\{a,b,c\}$.

We have $a\geq 1$ and $b+c\leq 2$.

Let $A:=b^2+c^2+3, B:=c^2+a^2+3, C:=a^2+b^2+3$.

After squaring, the desired inequality is equivalent to

$$2\sqrt{A}\left(\sqrt{B}+\sqrt{C}\right)+2\sqrt{BC}\geq 3(a^2+b^2+c^2+7)$$

$$2\sqrt{A\left(B+C+2\sqrt{BC}\right)}+2\sqrt{BC}\geq 3(a^2+b^2+c^2+7).$$

Now, we will prove that

$$\sqrt{BC}=\sqrt{(a^2+b^2+3)(a^2+c^2+3)}\geq a^2+3+\frac{b^2+c^2+bc}{3}.$$

After squaring, the inequality becomes

$$(3a^2+9-b^2-c^2-4bc)(b-c)^2\geq 0.$$

which is true since

$$3a^2+9-b^2-c^2-4bc\overset{a=3-b-c}{\triangleq}36-18(b+c)+2(b^2+c^2+bc)\geq$$

$$\geq 36-18(b+c)+\frac{3(b+c)^2}{2}=\frac{3}{2}(2-b-c)(10-b-c)+6\geq 0.$$

Therefore, it suffices to prove that

$$2\sqrt{A\left(B+C+2a^2+6+\frac{2(b^2+c^2+bc)}{3}\right)}\geq a^2+\frac{7(b^2+c^2)-2bc}{3}+15$$

$$\Leftrightarrow 12(b^2+c^2+3)(12a^2+36+5b^2+5c^2+2bc)\geq(3a^2+45+7b^2+7c^2-2bc)^2$$

Since $b^2+c^2=(b+c)^2-2bc$
$=a^2-6a+9-2bc$, **then the last inequality is equivalent to**

$$12(a^2-6a+12-2bc)(17a^2-30a+81-8bc)\geq(10a^2-42a+108-16bc)^2$$

$\Leftrightarrow f(bc)=104a^4-744a^3+1656a^2-1080a-(184a^2+48a-360)bc-64(bc)^2$
≥ 0. f **is concave and since**

$$bc\leq\frac{(b+c)^2}{4}=\frac{(3-a)^2}{4}$$

$$bc=(a-b)(a-c)+a(b+c-a)\geq a(3-2a),$$

So it suffices to prove that $f\left(\frac{(3-a)^2}{4}\right)\geq 0$ **and** $f\big(a(3-2a)\big)\geq 0$. **We have**

$$f\left(\frac{(3-a)^2}{4}\right)=54a^4-432a^3+1188a^2-1296a+486=54(a-1)^2(3-a)^2\geq 0.$$

$$f\big(a(3-2a)\big)=216a^4-432a^3+216a^2=216a^2(a-1)^2\geq 0.$$

Equality holds for $a=b=c=1$ & $a=3, b=c=0$ **and its permutation.**

1546. Let $a,b,c \geq 0, ab+bc+ca > 0$. Prove that :

$$\frac{2c^2+2ca+2cb-ab}{a^2+b^2}+\frac{2a^2+2ab+2ac-bc}{b^2+c^2}+\frac{2b^2+2bc+2ba-ca}{c^2+a^2} \geq \frac{15}{2}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

WLOG we assume that $a+b+c=1$. Let $p := a+b+c=1, q := ab+bc+ca, r := abc$.

$$\sum_{cyc}\frac{2a^2+2ab+2ac-bc}{b^2+c^2}=\sum_{cyc}\frac{2a-bc}{b^2+c^2}=\frac{\sum_{cyc}(2a-bc)[a^2(a^2+b^2+c^2)+b^2c^2]}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}$$

$$=\frac{2\sum_{cyc}a^3.\sum_{cyc}a^2+2abc\sum_{cyc}bc-abc\sum_{cyc}a^2-\sum_{cyc}(bc)^3}{\sum_{cyc}a^2.\sum_{cyc}(bc)^2-(abc)^2}$$

$$=\frac{2(1-3q+3r)(1-2q)+2qr-r(1-2q)-(q^3-3qr+3r^2)}{(1-2q)(q^2-2r)-r^2}\overset{?}{\geq}\frac{15}{2}$$

$$\Leftrightarrow f(r)=4-20q+9q^2+28q^3+(40-70q)r+9r^2 \geq 0.$$

Since $3q \leq p^2=1$, then $40-70q>0$, and

If $q \leq \frac{1}{4}$, we have : $f(r) \geq f(0)=(1-4q)(4-4q-7q^2) \geq 0$.

If $\frac{1}{4} \leq q \leq \frac{1}{3}$, by the fourth degree Schur's inequality, we have

$$f(r) \geq f\left(\frac{(p^2-q)(4q-p^2)}{6p}\right)=\frac{(4q-1)(1-3q)(29-67q-4q^2)}{12} \geq 0.$$

Equality holds iff $a=b=c$ & $a=b, c=0$ and its permutation.

1547. Let $a,b,c \geq 0, ab+bc+ca+abc=4$. Prove that :

$$\frac{a+b}{ab+2}+\frac{b+c}{bc+2}+\frac{c+a}{ca+2} \leq \frac{2}{3}(a+b+c)$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $p := a+b+c, q := ab+bc+ca, r := abc$. We have $4 = q+r \leq \frac{p^2}{3}+\frac{p^3}{27}$, then $p \geq 3$, and

$$\sum_{cyc}\frac{a+b}{ab+2} = \frac{\sum_{cyc}(a+b)(bc+2)(ca+2)}{(ab+2)(bc+2)(ca+2)} =$$

$$= \frac{2\sum_{cyc}a^2(b+c)+12abc+2abc\sum_{cyc}bc+8\sum_{cyc}a}{8+4\sum_{cyc}bc+2abc\sum_{cyc}a+(abc)^2} =$$

$$= \frac{2pq+6r+2qr+8p}{8+4q+2pr+r^2} \overset{q=4-r}{=} \frac{16p+2(7-p)r-2r^2}{24+2(p-2)r+r^2} \overset{?}{\leq} \frac{2p}{3}$$

$$\Leftrightarrow r[2p^2-p-21+(p+3)r] \geq 0.$$

If $p \geq 4$, we have $2p^2-p-21+(p+3)r \geq 2p^2-p-21 = (p-4)(2p+7)+7 \geq 0$.

If $3 \leq p \leq 4$, by Schur's inequality, we have $p^3+9r \geq 4pq = 4p(4-r)$,

then $r \geq \frac{16p-p^3}{4p+9}$, and

$$2p^2-p-21+(p+3)r \geq 2p^2-p-21+\frac{(p+3)(16p-p^3)}{4p+9} =$$

$$= \frac{(p-3)(63+36p+2p^2-p^3)}{4p+9} \geq 0$$

Equality holds iff $(a=b=c=1)$ and $(ab=4, c=0)$ and its permutation.

1548. If $a,b,c > 0$ and $a^4+b^4+c^4 = 3$ then:

$$\frac{a}{2+a}+\frac{b}{2+b}+\frac{c}{2+c} \leq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$3 = a^4 + b^4 + c^4 = \frac{a^4}{1^3} + \frac{b^4}{1^3} + \frac{c^4}{1^3} \overset{RADON}{\geq} \frac{(a+b+c)^4}{3^3}$$

$$3^4 \geq (a+b+c)^4 \text{ or } (a+b+c) \leq 3 \quad (1)$$

$$\frac{a}{2+a} + \frac{b}{2+b} + \frac{c}{2+c} = \sum \frac{a}{2+a} = \sum \left(1 - \frac{2}{2+a}\right) =$$

$$= 3 - 2\sum \frac{1}{2+a} \overset{Bergstrom}{\leq} 3 - 2 \times \frac{(1+1+1)^2}{6+(a+b+c)} \overset{(1)}{\leq} 3 - 2 \times \frac{9}{6+3} = 1$$

Equality holds for $a = b = c = 1$.

1549. If $a, b, c > 0, a^2 + b^2 + c^2 = 3$ then:

$$\frac{ab}{2+ab} + \frac{bc}{2+bc} + \frac{ca}{2+ca} \leq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{ab}{2+ab} + \frac{bc}{2+bc} + \frac{ca}{2+ca} = \sum \frac{ab}{2+ab} = \sum \left(1 - \frac{2}{2+ab}\right) =$$

$$= 3 - 2\sum \frac{1}{2+ab} \overset{Bergstrom}{\leq} 3 - 2 \times \frac{(1+1+1)^2}{6+ab+bc+ca} \leq$$

$$\leq 3 - 2 \times \frac{(1+1+1)^2}{6 + a^2 + b^2 + c^2} \overset{a^2+b^2+c^2=3}{=} 3 - 2 \times \frac{9}{9} = 1$$

Equality holds for $a = b = c = 1$.

1550. If $x \in \left[0, \frac{\pi}{2}\right]$, then prove that :

$$\sin x \leq \frac{4}{\pi}x - \frac{4}{\pi^2}x^2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } f(x) = \frac{\sin x}{x} + \frac{4x}{\pi^2} - \frac{4}{\pi} \ \forall x \in \left(0, \frac{\pi}{2}\right] \text{ and then :}$$

$$f'(x) = -\frac{\sin x}{x^2} + \frac{\cos x}{x} + \frac{4}{\pi^2} \text{ and } f''(x) = -\frac{(x^2-2)\sin x + 2x\cos x}{x^3}$$

$$\text{and now, let } F(x) = (x^2-2)\sin x + 2x\cos x \ \forall x \in \left[0, \frac{\pi}{2}\right] \text{ and then :}$$

$$F'(x) = x^2\cos x \geq 0 \ \forall x \in \left[0, \frac{\pi}{2}\right] \Rightarrow F(x) \text{ is } \uparrow \text{ on } \left[0, \frac{\pi}{2}\right] \Rightarrow F(x) \geq F(0) = 0,$$

$$'' ='' \text{ iff } x = 0 \Rightarrow (x^2-2)\sin x + 2x\cos x > 0 \ \forall x \in \left(0, \frac{\pi}{2}\right]$$

$$\Rightarrow f''(x) < 0 \ \forall x \in \left(0, \frac{\pi}{2}\right] \Rightarrow f'(x) \text{ is } \downarrow \text{ on } \left(0, \frac{\pi}{2}\right] \Rightarrow f'(x) \geq f'\left(\frac{\pi}{2}\right)$$

$$= -\frac{4}{\pi^2} + 0 + \frac{4}{\pi^2} = 0 \Rightarrow f'(x) \geq 0 \ \forall x \in \left(0, \frac{\pi}{2}\right] \Rightarrow f(x) \text{ is } \uparrow \text{ on } \left(0, \frac{\pi}{2}\right]$$

$$\Rightarrow f(x) \leq f\left(\frac{\pi}{2}\right) = \frac{2}{\pi} + \frac{4}{\pi^2}\cdot\frac{\pi}{2} - \frac{4}{\pi} = 0 \Rightarrow \frac{\sin x}{x} + \frac{4x}{\pi^2} - \frac{4}{\pi} \leq 0 \ \forall x \in \left(0, \frac{\pi}{2}\right]$$

$$\Rightarrow \sin x \leq \frac{4}{\pi}x - \frac{4}{\pi^2}x^2 \ \forall x \in \left(0, \frac{\pi}{2}\right] \text{ and } \because \sin(0) = \frac{4}{\pi}(0) - \frac{4}{\pi^2}(0)^2$$

$$\therefore \sin x \leq \frac{4}{\pi}x - \frac{4}{\pi^2}x^2 \ \forall x \in \left[0, \frac{\pi}{2}\right], '' ='' \text{ iff } x = 0 \text{ or } x = \frac{\pi}{2} \text{ (QED)}$$

1551. Let $a, b, c \geq 0, ab + bc + ca > 0$ and $a + b + c = 2.$ **Prove that :**

$$\frac{a\sqrt{bc+b+c}}{\sqrt{a}+\sqrt{bc}} + \frac{b\sqrt{ca+c+a}}{\sqrt{b}+\sqrt{ca}} + \frac{c\sqrt{ab+a+b}}{\sqrt{c}+\sqrt{ab}} \geq 2\left(ab+bc+ca-\sqrt{abc}\right)$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$4(bc+b+c) = (a+b+c)^2(bc+b+c) = [(b+c-a)^2 + 4a(b+c)](bc+b+c)$$
$$\geq \left[(b+c-a)\sqrt{bc} + 2\sqrt{a}(b+c)\right]^2.$$

Then $2\sqrt{bc+b+c} \geq (b+c-a)\sqrt{bc} + 2\sqrt{a}(b+c)$
$= 2\left(\sqrt{a}+\sqrt{bc}\right)(b+c) - (a+b+c)\sqrt{bc}$

$$\Rightarrow \sum_{cyc} \frac{a\sqrt{bc+b+c}}{\sqrt{a}+\sqrt{bc}} \geq \sum_{cyc} a\left(b+c-\frac{\sqrt{bc}}{\sqrt{a}+\sqrt{bc}}\right) = 2\sum_{cyc} bc - \sqrt{abc}.\sum_{cyc}\frac{\sqrt{a}}{\sqrt{a}+\sqrt{bc}}.$$

So it suffices to prove that

$$\sum_{cyc}\frac{\sqrt{a}}{\sqrt{a}+\sqrt{bc}}\le 2 \Leftrightarrow \sum_{cyc}\sqrt{a}(\sqrt{b}+\sqrt{ca})(\sqrt{c}+\sqrt{ab})$$
$$\le 2(\sqrt{a}+\sqrt{bc})(\sqrt{b}+\sqrt{ca})(\sqrt{c}+\sqrt{ab})$$

$$\Leftrightarrow 0\le\sqrt{abc}+2abc,$$

Equality holds iff $a=b=1, c=0$ and its permutation.

1552. **If $a,b,c>0$, then prove that :**

$$ab\sqrt{ab}+bc\sqrt{bc}+ca\sqrt{ca}\le\frac{1}{2}\cdot\sqrt[3]{\frac{(a^2+bc)^2(b^2+ca)^2(c^2+ab)^2}{abc}}+abc$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0,$
$y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$
$\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $=s,R,r$ (say); then : $\sum_{cyc}a=s\to(1), abc=r^2s\to(2),$

$$\sum_{cyc}ab=4Rr+r^2\to(3),\sum_{cyc}a^3=s^3-12Rrs\to(4)$$

We have : $ab\sqrt{ab}+bc\sqrt{bc}+ca\sqrt{ca}-abc\overset{AM-GM}{\le}\sum_{cyc}\frac{bc(b+c)}{2}-abc$

$$=\frac{(\sum_{cyc}a)(\sum_{cyc}ab)-5abc}{2}\overset{?}{\le}\frac{1}{2}\cdot\sqrt[3]{\frac{(a^2+bc)^2(b^2+ca)^2(c^2+ab)^2}{abc}}$$

$$=\frac{1}{2}\cdot\sqrt[3]{\frac{\left(abc\sum_{cyc}a^3+\sum_{cyc}a^3b^3+2a^2b^2c^2\right)^2}{abc}}\ \underset{\Leftrightarrow}{\text{via }(1),(2),(3)\text{ and }(4)}$$

$$r^2s(s(4Rr+r^2)-5r^2s)^3\overset{?}{\le}(r^2s(s^3-12Rrs)+(4Rr+r^2)^3-12Rr^3s^2+2r^4s^2)^2$$
$$\Leftrightarrow s^8-(48Rr-4r^2)s^6+r(64R^3+864R^2r-264Rr^2+70r^3)s^4-$$
$$r^2(3072R^4+2048R^3r+384R^2r^2-4r^4)s^2+r^2(4R+r)^6\overset{?}{\underset{(*)}{\ge}}0 \text{ and } \because P=$$

$$(s^2-16Rr+5r^2)^4+8r(2R-2r)(s^2-16Rr+5r^2)^3+$$
$$8r(8R^3+12R^2r-39Rr^2+20r^3)(s^2-16Rr+5r^2)^2-$$
$$16r^2(64R^4-280R^3r+420R^2r^2-260Rr^3+56r^4)(s^2-4R^2-4Rr-3r^2)$$
$$\overset{Gerretsen}{\ge}0\ \therefore \text{ in order to prove } (*), \text{it suffices to prove : LHS of } (*)\overset{?}{\ge}P$$

$$\Leftrightarrow 224t^5-1408t^4+3316t^3-3619t^2+1820t-332\overset{?}{\geq}0\left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)^2\left((t-2)(224t^2-64t+244)+105\right)\overset{?}{\geq}0\to \text{true} \because t\overset{\text{Euler}}{\geq}2$$

$$\Rightarrow (*)\text{ is true} \therefore \sum_{\text{cyc}} ab\sqrt{ab}\leq\frac{1}{2}\cdot\sqrt[3]{\frac{(a^2+bc)^2(b^2+ca)^2(c^2+ab)^2}{abc}}+abc$$

$$\forall a,b,c>0,''='' \text{ iff } a=b=c\ (\text{QED})$$

1553. If $x+z-yz=1$ and $y-3z+xz=1$, then prove that :

$$6-2\sqrt{5}\leq x^2+y^2\leq 6+2\sqrt{5}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Easy to see that if}: z=0, \text{then}: x=y=1 \text{ and then}:$$

$$6-2\sqrt{5}<x^2+y^2=2<6+2\sqrt{5} \text{ and we now focus on}: z\neq 0 \text{ and then}:$$

$$x\neq 1, y\neq 1 \text{ and } x+z-yz=1\Rightarrow z(1-y)\overset{(1)}{=}1-x \text{ and also}, y-3z+xz=1$$

$$\Rightarrow z(x-3)\overset{(2)}{=}1-y \text{ and } (1)\div(2)\Rightarrow\frac{1-y}{x-3}=\frac{1-x}{1-y}\Rightarrow 4x-x^2-3=y^2-2y+1$$

$$\Rightarrow x^2+y^2=2(2x+y)-4\overset{\text{CBS}}{\leq}2\sqrt{5(x^2+y^2)}-4$$

$$\Rightarrow t^2-2\sqrt{5}.t+4\overset{(*)}{\leq}0\ \left(t=\sqrt{x^2+y^2}\right)$$

$$\text{Now}, (*)\Rightarrow t\leq\frac{2\sqrt{5}+\sqrt{20-16}}{2}\Rightarrow\sqrt{x^2+y^2}\leq\sqrt{5}+1$$

$$\Rightarrow x^2+y^2\leq 6+2\sqrt{5}\to(\text{i})$$

$$\text{Also}, (*)\Rightarrow t\geq\frac{2\sqrt{5}-\sqrt{20-16}}{2}\Rightarrow\sqrt{x^2+y^2}\geq\sqrt{5}-1\Rightarrow x^2+y^2\geq 6-2\sqrt{5}$$

$$\to(\text{ii}) \therefore (\text{i}) \text{ and } (\text{ii})\Rightarrow 6-2\sqrt{5}\leq x^2+y^2\leq 6+2\sqrt{5}, \text{equalities occur when}:$$

$$x=2y\Rightarrow \text{when}: 5y^2-10y+4=0\Rightarrow \text{when}: y=\frac{5\pm\sqrt{5}}{5} \text{ and so},$$

$$\text{equality for}: 6-2\sqrt{5}\leq x^2+y^2 \text{ occurs when}: \begin{pmatrix} x=\dfrac{10-2\sqrt{5}}{5} \\ y=\dfrac{5-\sqrt{5}}{5}\end{pmatrix} \text{ and}$$

$$\text{equality for}: x^2+y^2\leq 6+2\sqrt{5} \text{ occurs when}: \begin{pmatrix} x=\dfrac{10+2\sqrt{5}}{5} \\ y=\dfrac{5+\sqrt{5}}{5}\end{pmatrix} (\text{QED})$$

1554. If $a,b,c>0, abc=1$ then:

$$(a+b)(b+c)(c+a)\ge(a+1)(b+1)(c+1)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$ab+bc+ca\overset{AM-GM}{\ge}3\sqrt[3]{a^2b^2c^2}\overset{abc=1}{=}3\ (1)$$
$$\frac{ab+bc+ca}{3}\ge 1\ (2)$$
$$\textit{We need to show:}$$
$$(a+b)(b+c)(c+a)\ge(a+1)(b+1)(c+1)$$
$$(a+b+c)(ab+bc+ca)-abc\ge 1+(a+b+c)+(ab+bc+ca)+abc$$
$$(a+b+c)(ab+bc+ca)\ge 1+(a+b+c)+(ab+bc+ca)+2abc$$
$$(a+b+c)(ab+bc+ca)\overset{abc=1}{\ge}1+(a+b+c)+(ab+bc+ca)+2$$
$$(a+b+c)(ab+bc+ca-1)\ge 3+(ab+bc+ca)$$
$$\sqrt{(a+b+c)^2}\left(\sum ab-\frac{\sum ab}{3}\right)\overset{(2)}{\ge}3+\sum ab$$
$$\sqrt{3\left(\sum ab\right)}\left(\sum ab-\frac{\sum ab}{3}\right)\ge 3+\sum ab$$
$$\frac{2}{\sqrt{3}}\left(\sum ab\right)^{\frac{3}{2}}\ge 3+\sum ab,$$
$$\frac{2}{\sqrt{3}}x^3\overset{let\ \sum ab=x^2}{\ge}3+x^2\ or,2x^3-\sqrt{3}x^2-3\sqrt{3}\ge 0$$
$$\left(x-\sqrt{3}\right)\left(2x^2+\sqrt{3}x+3\right)>0\ true\ as\ x^2=\sum ab\overset{(1)}{\ge}3$$

$$\textit{Equality holds for } a=b=c=1.$$

1555. If $a,b,c\in\mathbb{R}$ then prove that :

$$(a^3+b^3+c^3-3abc)^2\le(a^2+b^2+c^2)^3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(a^3+b^3+c^3-3abc)^2\le(a^2+b^2+c^2)^3$$
$$\Leftrightarrow\left(\sum_{cyc}a^2\right)^3\ge\left(\sum_{cyc}a\right)^2\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)^2\Leftrightarrow x^3\ge(x+2y)(x-y)^2$$

$$= x^3 - 3xy^2 + 2y^3\left(x = \sum_{cyc} a^2, y = \sum_{cyc} ab\right) \Leftrightarrow y^2\left(3\sum_{cyc} a^2 - 2\sum_{cyc} ab\right) \ge 0$$

$$\Leftrightarrow \left(\sum_{cyc} ab\right)^2\left(\sum_{cyc} a^2 + \sum_{cyc}(a-b)^2\right) \ge 0 \to \text{true}$$

$$\therefore (a^3 + b^3 + c^3 - 3abc)^2 \le (a^2 + b^2 + c^2)^3\ \forall a, b, c \in \mathbb{R},$$

$$'' ='' \text{ iff } a = b = c = 0\ (\text{QED})$$

1556. **If $a, b \in [0; 2]$ then prove that :**

$$\frac{8 + 6(a+b) + (a+b)^2}{4 + 2(a+b) + ab} \le 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{8 + 6(a+b) + (a+b)^2}{4 + 2(a+b) + ab} \le 3 \Leftrightarrow 4 - (a+b)^2 + 3ab \ge 0$$

$$\Leftrightarrow 4 - a^2 - b^2 + ab \overset{(*)}{\ge} 0$$

Case 1 $a \ge b$ and then $\because\ a \le 2 \wedge b \ge 0 \therefore (4 - a^2) - b^2 + ab \ge b(a-b) \ge 0$
$\Rightarrow (*)$ is true

Case 2 $b \ge a$ and then $\because\ b \le 2 \wedge a \ge 0 \therefore (4 - b^2) - a^2 + ab \ge a(b-a) \ge 0$

$$\Rightarrow (*) \text{ is true } \therefore \frac{8 + 6(a+b) + (a+b)^2}{4 + 2(a+b) + ab} \le 3\ \forall a, b \in [0; 2],$$

$$'' ='' \text{ iff } (a = 2, b = 0) \text{ or } (b = 2, a = 0)\ (\text{QED})$$

1557. **If $a, b, c, u, v, w > 0$, then :**

$$(a^2 + 2u^2)(b^2 + 2v^2)(c^2 + 2w^2) \ge 3(avw + buw + cuv)^2$$

Proposed by D.M. Bătinețu-Giurgiu, Dan Nănuți-Romania

Solution by Soumava Chakraborty-Kolkata-India

Due to Arkady Alt, if $x, y, z, T > 0$, then :

$$(x^2 + T^2)(y^2 + T^2)(z^2 + T^2) \ge \frac{3}{4}.T^4.(xy + yz + zx)^2,$$

$$\text{with equality iff } x=y=z=\frac{1}{\sqrt{2}}\to(1)$$

$$(a^2+2u^2)(b^2+2v^2)(c^2+2w^2)=$$

$$=\frac{(2a^2v^2w^2+4u^2v^2w^2)(2b^2u^2w^2+4u^2v^2w^2)(2c^2u^2v^2+4u^2v^2w^2)}{8(uvw)^4}=$$

$$=\frac{(x^2+T^2)(y^2+T^2)(z^2+T^2)}{8(uvw)^4}\ (x=\sqrt{2}.avw, y=\sqrt{2}.buw, z=\sqrt{2}.cuv, T=2uvw)\geq$$

$$\overset{\text{Arkady Alt}}{\geq}\frac{\frac{3}{4}.T^4.(xy+yz+zx)^2}{8(uvw)^4}=\frac{\frac{3}{4}.(2uvw)^4.\left(\sqrt{2}.avw+\sqrt{2}.buw+\sqrt{2}.cuv\right)^2}{8(uvw)^4}$$

$$\therefore (a^2+2u^2)(b^2+2v^2)(c^2+2w^2)\geq 3(avw+buw+cuv)^2\ \forall a,b,c,u,v,w>0,$$

$$''='' \text{ iff } avw=buw=cuv=\frac{1}{2}\ (\text{QED})$$

1558. If $a,b,c,s,t,u>0$, *then* :

$$(a^2+s^2)(b^2+t^2)(c^2+u^2)\geq\frac{3}{4}(atu+bsu+cst)^2$$

Proposed by D.M. Bătinețu-Giurgiu, Dan Nănuți-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Due to Arkady Alt, if } x,y,z,T>0, \textit{then}:$$

$$(x^2+T^2)(y^2+T^2)(z^2+T^2)\geq\frac{3}{4}.T^4.(xy+yz+zx)^2,$$

$$\text{with equality iff } x=y=z=\frac{1}{\sqrt{2}}\to(1)$$

$$(a^2+s^2)(b^2+t^2)(c^2+u^2)=$$

$$=\frac{(a^2t^2u^2+s^2t^2u^2)(b^2s^2u^2+s^2t^2u^2)(c^2s^2t^2+s^2t^2u^2)}{(stu)^4}=$$

$$=\frac{(x^2+T^2)(y^2+T^2)(z^2+T^2)}{(stu)^4}\ (x=atu, y=bsu, z=cst, T=stu)\geq$$

$$\overset{\text{Arkady Alt}}{\geq}\frac{\frac{3}{4}.T^4.(xy+yz+zx)^2}{(stu)^4}=\frac{\frac{3}{4}.(stu)^4.(atu+bsu+cst)^2}{(stu)^4}$$

$$\therefore (a^2+s^2)(b^2+t^2)(c^2+u^2)\ge \frac{3}{4}(atu+bsu+cst)^2\ \forall\, a,b,c,s,t,u>0,$$
$$''='' \text{ iff } atu=bsu=cst=\frac{1}{\sqrt{2}}\ (\text{QED})$$

1559. If $a,b,c>0,\ a+b+c=3$ then:

$$\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}+1\ge\frac{8}{3}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ x=ab+bc+ca, y=abc$$
$$then\ y=abc\overset{AM-GM}{\le}\left(\frac{a+b+c}{3}\right)^3\overset{a+b+c=3}{=}1\ (1)$$

$$x=ab+bc+ca\overset{AM-GM}{\ge}3(abc)^{\frac{2}{3}}=3y^{\frac{2}{3}}\ (2)$$

$$x-3y\overset{(2)}{\ge}3y^{\frac{2}{3}}-3y=3y^{\frac{2}{3}}(1-y)>0\ as\ from\ (1),0<y<1\ (3)$$

$$2x-y-9=2(ab+bc+ca)-abc-3^2=$$
$$\overset{a+b+c=3}{=}2(ab+bc+ca)-abc-(a+b+c)^2=-(a^2+b^2+c^2+abc)<0\ (4)$$
$$a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)\overset{a+b+c=3}{=}9-2x\ (5)$$

$$\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}+1=\frac{a^2+b^2+c^2}{abc}+1=\frac{9-2x}{y}+1=\frac{9-2x+y}{y}$$

$$\frac{8}{3}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)=\frac{8}{3}\frac{(a+b+c)(a^2+b^2+c^2)+3abc}{(a+b+c)(ab+bc+ca)-abc}=$$

$$\overset{(5)\ \&\ a+b+c=3}{=}\frac{8}{3}\frac{3(9-2x)+3y}{3x-y}=\frac{72-16x+y}{3x-y}$$

$$We\ need\ to\ show,$$
$$\frac{9-2x+y}{y}\ge\frac{72-16x+y}{3x-y}$$
$$(9-2x+y)(3x-y)\ge 72y-16xy+y^2$$
$$2x^2-7xy+3y^2-9x+27y\le 0$$

$$(x-3y)(2x-y)-9(x-3y)\le 0$$
$$(x-3y)(2x-y-9)\le 0 \text{ true by } (3)\ \&\ (4)$$

Equality holds for $a=b=c=1$.

1560. If $a,b,c>0$ then:

$$a^3+b^3+c^3+\frac{a}{(b+c)^2}+\frac{b}{(a+c)^2}+\frac{c}{(b+a)^2}\ge (a+b+c)$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$a^3+b^3+c^3+\frac{a}{(b+c)^2}+\frac{b}{(a+c)^2}+\frac{c}{(b+a)^2}=$$
$$=a^3+b^3+c^3+\frac{a^3}{(ab+ac)^2}+\frac{b^3}{(bc+ab)^2}+\frac{c^3}{(ac+bc)^2}=$$
$$=a^3\left(1+\frac{1}{(ab+ac)^2}\right)+b^3\left(1+\frac{1}{(bc+ab)^2}\right)+c^3\left(1+\frac{1}{(ac+bc)^2}\right)\overset{A-G}{\ge}$$
$$\ge a^3.\frac{2}{(ab+ac)}+b^3.\frac{2}{(bc+ab)}+c^3.\frac{2}{(ac+bc)}=$$
$$=2\left(\frac{a^2}{b+c}+\frac{b^2}{a+c}+\frac{c^2}{a+b}\right)\overset{Bergstrom}{\ge}2.\frac{(a+b+c)^2}{2(a+b+c)}=a+b+c$$

$$Equality\ \ holds\ \ for\ \ a=b=c=\frac{1}{\sqrt{2}}$$

1561. If $a,b>0$ then prove that :

$$\frac{4(a^4+b^4)}{a^2+b^2}-\frac{(a+b)^2}{2}+\frac{1}{a^2}+\frac{1}{b^2}\ge 4$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{4(a^4+b^4)}{a^2+b^2}-\frac{(a+b)^2}{2}+\frac{1}{a^2}+\frac{1}{b^2}\ge\frac{2(a^2+b^2)^2}{a^2+b^2}-\frac{(a+b)^2}{2}+\frac{1}{a^2}+\frac{1}{b^2}\ge$$

$$\ge(a+b)^2-\frac{(a+b)^2}{2}+\frac{1^3}{a^2}+\frac{1^3}{b^2}\overset{Radon}{\ge}\frac{(a+b)^2}{2}+\frac{(1+1)^3}{(a+b)^2}=\frac{(a+b)^2}{2}+\frac{8}{(a+b)^2}\ge$$

$$\overset{AM-GM}{\geq} 2\cdot\sqrt{\frac{(a+b)^2}{2}\cdot\frac{8}{(a+b)^2}} \Rightarrow \frac{4(a^4+b^4)}{a^2+b^2}-\frac{(a+b)^2}{2}+\frac{1}{a^2}+\frac{1}{b^2}\geq 4\ \forall\ a,b>0,$$

$$''='' \text{ iff } a=b=2 \text{ (QED)}$$

1562. **If $a,b>0$ and $a^2+b^2\leq\frac{2}{9}$ then prove that :**

$$2\ln(a+b)-\frac{1}{2}\ln\left(2(a^2+b^2)\right)-\frac{1}{4}\cdot\sqrt{2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}\cdot(a+b)^2\leq\ln\frac{2}{3}-\frac{2}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$2\ln(a+b)-\frac{1}{2}\ln\left(2(a^2+b^2)\right)-\frac{1}{4}\cdot\sqrt{2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}\cdot(a+b)^2$$

$$=\ln\frac{(a+b)^2}{\sqrt{2a^2+2b^2}}-\frac{1}{4}\cdot\sqrt{2\left(\frac{1^2}{a^2}+\frac{1^2}{b^2}\right)}\cdot(a+b)^2 \overset{Bergstrom}{\leq} \ln\frac{(a+b)^2}{\sqrt{2a^2+2b^2}}-$$

$$\frac{1}{4}\cdot\sqrt{\frac{16}{2a^2+2b^2}}\cdot(a+b)^2 \therefore \text{LHS}\leq\ln x-x\rightarrow (1)\ \left(x=\frac{(a+b)^2}{\sqrt{2a^2+2b^2}}\right)$$

$$\text{Now, } x=\frac{(a+b)^2}{\sqrt{2a^2+2b^2}}\overset{CBS}{\leq}\frac{2a^2+2b^2}{\sqrt{2a^2+2b^2}}=\sqrt{2(a^2+b^2)}\overset{a^2+b^2\leq\frac{2}{9}}{\leq}\frac{2}{3} \text{ and if } f(x)=$$

$$\ln x-x\ \forall\ x\in\left(0,\frac{2}{3}\right], \text{ then : } f'(x)=\frac{1-x}{x}>0\Rightarrow f(x) \text{ is } \uparrow \text{ on } \left(0,\frac{2}{3}\right]$$

$$\Rightarrow f(x)\leq f\left(\frac{2}{3}\right)\ \left(\because x\leq\frac{2}{3}\right)\Rightarrow\ln x-x\leq\ln\frac{2}{3}-\frac{2}{3}\overset{via\ (1)}{\Rightarrow}$$

$$2\ln(a+b)-\frac{1}{2}\ln\left(2(a^2+b^2)\right)-\frac{1}{4}\cdot\sqrt{2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}\cdot(a+b)^2\leq\ln\frac{2}{3}-\frac{2}{3}$$

$$\forall\ a,b>0 \mid a^2+b^2\leq\frac{2}{9}, ''='' \text{ iff } a=b=\frac{1}{3} \text{ (QED)}$$

1563. If $a,b,c\ge 0$ and $a+b+c=\sqrt{3}$ then prove that :

$$-2\left(\sum_{cyc}ab\right)^3+27a^2b^2c^2-3\sum_{cyc}a^2+6\left(\sum_{cyc}ab\right)\le 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

When exactly two variables $=0$ and WLOG we may assume $b=c=0$ $(a=\sqrt{3})$, then : $LHS=-3a^2<0<2$ and when exactly one variable $=0$ & WLOG we may assume $a=0$ $(b,c>0)$, $LHS=-2b^3c^3-3(b^2+c^2)+6bc$ $=-2b^3c^3-3(b-c)^2<0<2$ and we now consider : $a,b,c>0$ and assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0$, $y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$ $\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $=s,R,r$ (say); then : $\sum_{cyc}a=s\to(1)$, $abc=r^2s\to(2)$, $\sum_{cyc}ab=4Rr+r^2\to(3)$,

$\sum_{cyc}a^2=s^2-8Rr-2r^2\to(4)$ and so, $2+2\left(\sum_{cyc}ab\right)^3+3\sum_{cyc}a^2-6\left(\sum_{cyc}ab\right)-$

$$27a^2b^2c^2\overset{a+b+c=\sqrt{3}}{=}\frac{2}{27}\left(\sum_{cyc}a\right)^6+2\left(\sum_{cyc}ab\right)^3+\frac{3}{9}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a\right)^4-$$

$$\frac{6}{9}\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)^4-27a^2b^2c^2\overset{?}{\ge}0\overset{\text{via }(1),(2),(3)\text{ and }(4)}{\Leftrightarrow}$$

$$2s^6+54(4Rr+r^2)^3+9s^4(s^2-8Rr-2r^2)-18(4Rr+r^2)s^4-729r^4s^2\overset{?}{\ge}0$$

$$\Leftrightarrow 11s^6-(144Rr+36r^2)s^4-729r^4s^2+54r^3(4R+r)^3\overset{(*)}{\ge}0\text{ and }\because P=$$

$$11(s^2-16Rr+5r^2)^3+(384Rr-201r^2)(s^2-16Rr+5r^2)^2+$$

$$8r^2(480R^2-624Rr+57r^2)(s^2-16Rr+5r^2)\overset{\text{Gerretsen}}{\geq}0\therefore\text{ to prove }(*),\text{it}$$

$$\text{suffices to prove : LHS of }(*)\overset{?}{\geq}P\Leftrightarrow 1456t^3-3228t^2+543t+178\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(1456t^2-316t-89)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow(*)\text{ is true}$$

$$\therefore -2\left(\sum_{cyc}ab\right)^3+27a^2b^2c^2-3\sum_{cyc}a^2+6\left(\sum_{cyc}ab\right)\leq 2\ \forall\, a,b,c\geq 0,$$

$$''='' \text{ iff } a=b=c=\frac{1}{\sqrt{3}}\ (\text{QED})$$

1564. **If $a,b,c>0$ and $(a+b)^2+2c^2\geq 2$ then prove that :**

$$5(a^2+b^2+c^2)-\left(a+b+\sqrt{2}c\right)^2-\sqrt{\frac{(a+b)^2}{2}+c^2}\geq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$5(a^2+b^2+c^2)-\left(a+b+\sqrt{2}c\right)^2-\sqrt{\frac{(a+b)^2}{2}+c^2}\overset{\text{CBS}}{\geq}$$

$$5\left(\frac{(a+b)^2}{2}+c^2\right)-2((a+b)^2+2c^2)-\sqrt{\frac{(a+b)^2+2c^2}{2}}=\frac{5x}{2}-2x-\sqrt{\frac{x}{2}}$$

$$(x=(a+b)^2+2c^2)=\sqrt{\frac{x}{2}}.\left(\sqrt{\frac{x}{2}}-1\right)=\sqrt{\frac{x}{2}}.\frac{\frac{x}{2}-1}{\sqrt{\frac{x}{2}}+1}\geq 0\because x\geq 2\Rightarrow\frac{x}{2}-1\geq 0$$

$$\therefore 5(a^2+b^2+c^2)-\left(a+b+\sqrt{2}c\right)^2-\sqrt{\frac{(a+b)^2}{2}+c^2}\geq 0\ \forall\, a,b,c>0$$

$$\text{with }(a+b)^2+2c^2\geq 2,''='' \text{ iff } a=b\ \wedge\ a+b=\sqrt{2}c\ \wedge\ (a+b)^2+2c^2=2$$

$$\Rightarrow ''='' \text{ iff }\left(a=b=\frac{1}{2},c=\frac{1}{\sqrt{2}}\right)\ (\text{QED})$$

1565. **If $xz+yz+2y=1$ and $2xz+yz+x+3y=2$**

then prove that : $0\leq xy(1+z)\leq\frac{1}{4}$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(2xz+yz+x+3y)-(xz+yz+2y)=2-1\Rightarrow xz+x+y=1$$

$$\Rightarrow x(1+z)=1-y\Rightarrow xy(1+z)=y(1-y)=y-y^2-\frac{1}{4}+\frac{1}{4}=\frac{1}{4}-\frac{4y^2-4y+1}{4}$$

$$=\frac{1}{4}-\frac{(2y-1)^2}{4}\le\frac{1}{4}$$

$$\text{Now, }2(xz+yz+2y)=2xz+yz+x+3y\Rightarrow y+yz=x\Rightarrow y(1+z)=x$$

$$\Rightarrow xy(1+z)=x^2\ge 0 \text{ and so, } 0\le xy(1+z)\le\frac{1}{4} \text{ whenever } xz+yz+2y=1$$

$$\text{and } 2xz+yz+x+3y=2,''='' \text{ for LH inequality iff } x=0$$

$$\text{and }''=''\text{ for RH inequality iff } y=\frac{1}{2}\ (\text{QED})$$

1566.

$$\text{Let be } f:\ R_+^*\times R_+^*\to R_+^*,\quad \text{then}:$$

$$\left(\frac{x^2}{\left(y+f(x,y)\right)^2}+2\right).\left(\frac{y^2}{\left(y+f(x,y)\right)^2}+2\right).\left(\frac{\left(f(x,y)\right)^2}{(y+x)^2}+2\right)\ge\frac{27}{4}\quad \forall x,y\in R_+^*$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\text{According to}:$$

$$\boxed{\forall x,y,z,t>0\ \ (x^2+t^2)(y^2+t^2)(z^2+t^2)\ge\frac{3}{4}t^4(x+y+z)^2}$$

formula (*Arkady Alt*)

$$\frac{x}{y+f(x,y)}=a,\qquad \frac{y}{x+f(x,y)}=b,\qquad \frac{f(x,y)}{x+y}=c,\qquad t^2=2$$

$$\text{Then}:\ (a^2+2)(b^2+2)(c^2+2)\ge\frac{3}{4}.4(a+b+c)^2=$$

$$=3\left(\frac{x}{y+f(x,y)}+\frac{y}{x+f(x,y)}+\frac{f(x,y)}{x+y}\right)^2=$$

$$=3\left(\frac{x^2}{xy+xf(x,y)}+\frac{y^2}{xy+yf(x,y)}+\frac{f^2(x,y)}{f(x,y)(x+y)}\right)^2\overset{Bergstrom}{\ge}$$

$$\ge 3\left(\frac{\left(x+y+f(x,y)\right)^2}{2(xy+xf+yf)}\right)^2=\frac{3}{4}\left(\frac{(x+y+f)^2}{(xy+xf+yf)}\right)^2\ge\frac{3}{4}.3^2=\frac{27}{4}$$

1567. If $a,b,c>0, a+b+c=3abc$ then:

$$\frac{1}{\sqrt{3+a^2}}+\frac{1}{\sqrt{3+a^2}}+\frac{1}{\sqrt{3+a^2}}\le\frac{3}{2}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Tapas Das-India

$$a+b+c=3abc$$

$$3=\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}=\frac{1^2}{bc}+\frac{1^2}{ca}+\frac{1^2}{ab}\overset{Bergstrom}{\geq}\frac{(1+1+1)^2}{ab+bc+ca}=\frac{9}{ab+bc+ca}$$

$$3(ab+bc+ca)\geq 9 \text{ or } ab+bc+ca\geq 3 \ (1)$$

$$\sum\frac{1}{a+1}=\frac{(1+b)(1+c)+(1+c)(1+a)+(1+a)(1+b)}{(1+a)(1+b)(1+c)}=$$
$$=\frac{3+2(a+b+c)+(ab+bc+ca)}{1+(a+b+c)+(ab+bc+ca)+abc}$$

We will show:

$$\sum\frac{1}{a+1}\leq\frac{3}{2} \text{ or } \frac{3+2(a+b+c)+(ab+bc+ca)}{1+(a+b+c)+(ab+bc+ca)+abc}\leq\frac{3}{2}$$

$$6+4(a+b+c)+2(ab+bc+ca)\leq 3+3(a+b+c)+3(ab+bc+ca)+3abc$$

$$3+(a+b+c)\leq(ab+bc+ca)+3abc$$

$$ab+bc+ca\overset{a+b+c=3abc}{\geq}3 \text{ true by } (1)$$

$$\text{Therefore: }\sum\frac{1}{a+1}\leq\frac{3}{2} \ (2)$$

$$\frac{1}{\sqrt{3+a^2}}+\frac{1}{\sqrt{3+a^2}}+\frac{1}{\sqrt{3+a^2}}=\frac{1}{\sqrt{2+1+a^2}}+\frac{1}{\sqrt{2+1+a^2}}+\frac{1}{\sqrt{2+1+a^2}}\overset{AM-GM}{\leq}$$
$$\leq\frac{1}{\sqrt{2+2a}}+\frac{1}{\sqrt{2+2b}}+\frac{1}{\sqrt{2+2c}}\overset{CBS}{\leq}\sqrt{3\sum\frac{1}{2+2a}}.=\sqrt{\frac{3}{2}\sum\frac{1}{1+a}}.\overset{(2)}{\leq}\sqrt{\frac{3}{2}.\frac{3}{2}}=\frac{3}{2}$$

Equality holds for $a=b=c=1$.

1568. *If* $a,b,c>0,\ abc=1$ *and* $\lambda\geq 0$ *then* :

$$\sum\frac{a^3}{\sqrt{a^2+2bc+\lambda a}}\geq\frac{3}{\sqrt{\lambda+3}}$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum\frac{a^3}{\sqrt{a^2+2bc+\lambda a}}=\sum\frac{(a^2)^{\frac{3}{2}}}{\sqrt{a^2+2bc+\lambda a}}\overset{Radon}{\geq}$$

$$\geq\frac{\left(\sum_{cyc}a^2\right)^{\frac{3}{2}}}{\left(\sum_{cyc}a^2+2\sum_{cyc}bc+\lambda\sum_{cyc}a\right)^{\frac{1}{2}}}\overset{(1)}{\geq}\frac{\left(\sum_{cyc}a^2\right)^{\frac{3}{2}}}{\left(\sum_{cyc}a^2+2\sum_{cyc}a^2+\lambda\sum_{cyc}a^2\right)^{\frac{1}{2}}}$$

$$(1)\ \boxed{ab+bc+ac\leq a^2+b^2+c^2}\ (true)$$

$$\boxed{\begin{array}{c}a>0,\ \ b>0,\ \ c>0\ \ abc=1\\ a+b+c\leq a^2+b^2+c^2\end{array}}\ (true)$$

$$\frac{\left(\sum_{cyc}a^2\right)^{\frac{3}{2}}}{\left(\sum_{cyc}a^2\right)^{\frac{1}{2}}(3+\lambda)^{\frac{1}{2}}}=\frac{\sum_{cyc}a^2}{\sqrt{3+\lambda}}\overset{A-G}{\geq}\frac{3\left(\prod_{cyc}a^2\right)^{\frac{1}{3}}}{\sqrt{3+\lambda}}=\frac{3}{\sqrt{3+\lambda}}$$

$$Equality\ \ holds\ \ iff\ \ a=b=c=1$$

1569. **If $a,b,c>0$ and $abc=1$, then prove that :**

$$5+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq(1+a)(1+b)(1+c)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$5+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq(1+a)(1+b)(1+c)$$

$$\Leftrightarrow 5+\sum_{cyc}\frac{a}{b}\geq 1+abc+\sum_{cyc}a+\sum_{cyc}ab\overset{abc=1}{\Leftrightarrow}\sum_{cyc}\frac{a}{b}+3\geq\sum_{cyc}a+\sum_{cyc}ab$$

$$\Leftrightarrow\sum_{ccyc}\frac{xz}{y^2}+3\geq\sum_{cyc}\frac{x}{y}+\sum_{cyc}\frac{y}{x}\left(a=\frac{x}{y},b=\frac{y}{z},c=\frac{z}{x}\right)$$

$$\Leftrightarrow\sum_{cyc}x^3y^3+3x^2y^2z^2\geq xyz\left(\sum_{cyc}x^2y+\sum_{cyc}xy^2\right)\to\text{true via }3^{rd}-\text{degree Schur}$$

$$\text{with variables }xy,yz,zx\ \therefore\ 5+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq(1+a)(1+b)(1+c)$$

$$\forall\ a,b,c>0\text{ and }abc=1,''=''\text{ iff }a=b=c=1\ (QED)$$

1570. **If $(1+a^2b)(1+b^2c)(1+c^2a)\geq 8$ then prove that :**

$$a^2+b^2+c^2\geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$8 \le (1+a^2b)(1+b^2c)(1+c^2a) \Rightarrow 64 \le (1+a^2b)^2(1+b^2c)^2(1+c^2a)^2$$

$$\overset{CBS}{\le} (1+a^2b^2)(1+a^2).(1+b^2c^2)(1+b^2).(1+c^2a^2)(1+c^2)$$

$$\Rightarrow 2 \le \sqrt[6]{(1+a^2b^2)(1+b^2c^2)(1+c^2a^2)(1+a^2)(1+b^2)(1+c^2)}$$

$$\overset{AM-GM}{\le} \frac{6+\sum_{cyc}a^2b^2+\sum_{cyc}a^2}{6} \le \frac{6+\frac{\left(\sum_{cyc}a^2\right)^2}{3}+\sum_{cyc}a^2}{6}$$

$$\Rightarrow \left(\sum_{cyc}a^2\right)^2+3\left(\sum_{cyc}a^2\right)-18 \ge 0 \Rightarrow \left(\sum_{cyc}a^2-3\right)\left(\sum_{cyc}a^2+6\right) \ge 0$$

$$\Rightarrow \sum_{cyc}a^2-3 \therefore a^2+b^2+c^2 \ge 3 \text{ whenever } (1+a^2b)(1+b^2c)(1+c^2a) \ge 8,$$

$$'' ='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1571. If $a,b,c>0$, *then prove that*:

$$\sum_{cyc}\frac{\sqrt{(b^2+c^2)(c+a)}+\sqrt{(c^2+a^2)(b+c)}}{\sqrt{(a+b)(c+a)}+\sqrt{(a+b)(b+c)}} \ge \frac{3\sqrt{3}.(abc)^{\frac{1}{3}}}{\sqrt{a+b+c}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\forall A', B', C', x, y, z > 0, \frac{x}{y+z}(B'+C')+\frac{y}{z+x}(C'+A')+\frac{z}{x+y}(A'+B') \ge$$

$$\sqrt{3\sum_{cyc}A'B'} \text{ (via Walter Janous)} \to (1)$$

$$\sum_{cyc}\frac{\sqrt{(b^2+c^2)(c+a)}+\sqrt{(c^2+a^2)(b+c)}}{\sqrt{(a+b)(c+a)}+\sqrt{(a+b)(b+c)}} = \sum_{cyc}\frac{\frac{1}{\sqrt{a+b}}.\left(\sqrt{\frac{b^2+c^2}{b+c}}+\sqrt{\frac{c^2+a^2}{c+a}}\right)}{\frac{1}{\sqrt{b+c}}+\frac{1}{\sqrt{c+a}}}$$

$$= \frac{x}{y+z}(B'+C')+\frac{y}{z+x}(C'+A')+\frac{z}{x+y}(A'+B')$$

$$\begin{pmatrix} x=\frac{1}{\sqrt{a+b}}, y=\frac{1}{\sqrt{b+c}}, z=\frac{1}{\sqrt{c+a}}, \\ A'=\sqrt{\frac{a^2+b^2}{a+b}}, B'=\sqrt{\frac{b^2+c^2}{b+c}}, C'=\sqrt{\frac{c^2+a^2}{c+a}} \end{pmatrix} \overset{via (1)}{\ge} \sqrt{3\sum_{cyc}\left(\sqrt{\frac{a^2+b^2}{a+b}}.\sqrt{\frac{b^2+c^2}{b+c}}\right)}$$

$$\overset{AM-GM}{\geq} 3.\sqrt{\sqrt[3]{\frac{\prod_{cyc}(b^2+c^2)}{\prod_{cyc}(b+c)}}} \geq 3.\sqrt[6]{\frac{\frac{1}{8}\prod_{cyc}(b+c)^2}{\prod_{cyc}(b+c)}} \overset{Cesaro}{\geq} 3.\sqrt[6]{\frac{1}{8}.8abc} \overset{?}{\geq} \frac{3\sqrt{3}.(abc)^{\frac{1}{3}}}{\sqrt{a+b+c}}$$

$$\Leftrightarrow \sqrt{\frac{a+b+c}{3}} \overset{?}{\geq} (abc)^{\frac{1}{6}} \Leftrightarrow \sum_{cyc} a \overset{?}{\geq} 3.\sqrt[3]{abc} \to \text{true via } AM-GM$$

$$\therefore \sum_{cyc} \frac{\sqrt{(b^2+c^2)(c+a)}+\sqrt{(c^2+a^2)(b+c)}}{\sqrt{(a+b)(c+a)}+\sqrt{(a+b)(b+c)}} \geq \frac{3\sqrt{3}.(abc)^{\frac{1}{3}}}{\sqrt{a+b+c}} \ \forall a,b,c>0,$$

$$'' ='' \text{ iff } a=b=c \text{ (QED)}$$

Solution 2 by Mirsadix Muzefferov-Azerbaijan

$$LHS_1 = \frac{\sqrt{(c+a)(2b^2+2c^2)}+\sqrt{(c+a)(2b^2+2c^2)}}{\sqrt{2}\left(\sqrt{(a+b)(c+a)}+\sqrt{(a+b)(b+c)}\right)} \overset{A-G}{\geq}$$

$$\frac{\sqrt{(b+c)(c+a)}\left(\sqrt{b+c}+\sqrt{c+a}\right)}{\sqrt{2}\sqrt{a+b}\left(\sqrt{b+c}+\sqrt{c+a}\right)} = \frac{\sqrt{(b+c)(c+a)}}{\sqrt{2}\sqrt{a+b}}$$

Analogously , we obtain for the others.

$$LHS_2 \geq \frac{\sqrt{(b+a)(c+a)}}{\sqrt{2}\sqrt{c+b}};\ LHS_3 \geq \frac{\sqrt{(b+a)(c+b)}}{\sqrt{2}\sqrt{c+a}}$$

$$\sum_1^3 LHS_k \geq \frac{\sqrt{(b+c)(c+a)}}{\sqrt{2}\sqrt{a+b}}+\frac{\sqrt{(b+a)(c+a)}}{\sqrt{2}\sqrt{c+b}}+\frac{\sqrt{(b+a)(c+b)}}{\sqrt{2}\sqrt{c+a}} =$$

$$\frac{(b+c)(c+a)+(b+a)(c+a)+(b+a)(c+b)}{\sqrt{2}\sqrt{(a+b)(c+b)(c+a)}} = \frac{\left(a^2+b^2+c^2\right)+3(ab+bc+ac)}{\sqrt{2}\sqrt{(a+b)(c+b)(c+a)}} \quad (*)$$

$$\frac{(a+b)+(b+c)+(a+c)}{3} \geq \sqrt[3]{(a+b)(c+b)(c+a)} \to$$

$$\sqrt[3]{(a+b)(c+b)(c+a)} \leq \frac{2(a+b+c)}{3}$$

$$\left(\sqrt[3]{(a+b)(c+b)(c+a)}\right)^3 \leq \left(\frac{2(a+b+c)}{3}\right)^3 \to$$

$$\sqrt{(a+b)(c+b)(c+a)} \leq \frac{2\sqrt{2}(a+b+c)\sqrt{a+b+c}}{3\sqrt{3}} \to$$

$$\frac{1}{\sqrt{(a+b)(c+b)(c+a)}} \geq \frac{3\sqrt{3}}{2\sqrt{2}(a+b+c)\sqrt{a+b+c}} \quad (1)$$

Let's use the expression (1) *in* (*)

$$LHS \geq \frac{\left(a^2+b^2+c^2\right)+3(ab+bc+ac)}{\sqrt{2}\sqrt{(a+b)(c+b)(c+a)}} \overset{(1)}{\geq} \frac{3\sqrt{3}\left(\left(a^2+b^2+c^2\right)+3(ab+bc+ac)\right)}{4(a+b+c)\sqrt{a+b+c}} =$$

$$\frac{3\sqrt{3}}{\sqrt{a+b+c}}\cdot\frac{(a^2+b^2+c^2)+3(ab+bc+ac)}{4(a+b+c)}\geq\frac{3\sqrt{3}(abc)^{\frac{1}{3}}}{\sqrt{a+b+c}}$$

$$Equality\ holds\ for:\ a=b=c$$

1572. If $a,b,c>0$, *then prove that*:

$$\sum_{cyc}\frac{\pi^{a+b}.e^{a+b}+\pi^{2c}.e^{a+c}}{\pi^{a+b}.e^{b}+\pi^{b+c}.e^{c}}\geq 3e^{\frac{a+b+c}{3}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall\ A',B',C',x',y',z'>0,$$

$$\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')\geq\sqrt{3\sum_{cyc}A'B'}$$

$$(\text{via Walter Janous})\to① \text{ and } \sum_{cyc}\frac{\pi^{a+b}.e^{a+b}+\pi^{2c}.e^{a+c}}{\pi^{a+b}.e^{b}+\pi^{b+c}.e^{c}}=\sum_{cyc}\frac{xyXY+z^2XZ}{xyY+yzZ}$$

$$\binom{x=\pi^a,y=\pi^b,z=\pi^c,}{X=e^a,Y=e^b,Z=e^c}=\sum_{cyc}\frac{\frac{XY}{z}+\frac{zXZ}{xy}}{\frac{Y}{z}+\frac{Z}{x}}=\sum_{cyc}\frac{\frac{X}{y}\left(\frac{Yy}{z}+\frac{Zz}{x}\right)}{\frac{Y}{z}+\frac{Z}{x}}$$

$$=\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')$$

$$\left(x'=\frac{X}{y},y'=\frac{Y}{z},z'=\frac{Z}{x},A'=\frac{Xx}{y},B'=\frac{Yy}{z},C'=\frac{Zz}{x}\right)\overset{\text{via}①}{\geq}\sqrt{3\sum_{cyc}\left(\frac{Xx}{y}.\frac{Yy}{z}\right)}$$

$$\overset{AM-GM}{\geq}3.\sqrt{\sqrt[3]{X^2Y^2Z^2}}=3.\sqrt[3]{e^{a+b+c}}\therefore\sum_{cyc}\frac{\pi^{a+b}.e^{a+b}+\pi^{2c}.e^{a+c}}{\pi^{a+b}.e^{b}+\pi^{b+c}.e^{c}}\geq 3e^{\frac{a+b+c}{3}}$$

$$\forall\ a,b,c>0,''=''\ \text{iff}\ a=b=c\ (\text{QED})$$

1573. If $a,b,c>0$ *and* $0\leq\lambda\leq 1$ then:

$$\sum_{cyc}\frac{a^2}{b}\geq\lambda.\frac{\sum_{cyc}a^3}{\sum_{cyc}ab}+\left(1-\frac{\lambda}{3}\right)\sum_{cyc}a$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^2}{b}\ge\lambda.\frac{\sum_{cyc}a^3}{\sum_{cyc}ab}+\left(1-\frac{\lambda}{3}\right)\sum_{cyc}a\Leftrightarrow\frac{\sum_{cyc}ab^3}{abc}-\sum_{cyc}a\ge$$

$$\ge\lambda\left(\frac{\sum_{cyc}a^3}{\sum_{cyc}ab}-\frac{\sum_{cyc}a}{3}\right)\Leftrightarrow 3\left(\left(\sum_{cyc}ab^3\right)-abc\left(\sum_{cyc}a\right)\right)\left(\sum_{cyc}ab\right)\overset{(*)}{\ge}$$

$$\ge\lambda.abc\left(3\left(\sum_{cyc}a^3\right)-\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)\right)$$

$$3\left(\sum_{cyc}a^3\right)\overset{\text{Chebyshev}}{\ge}\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2\right)\ge\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)\ \&\because 0\le\lambda\le 1$$

$$\therefore\lambda.abc\left(3\left(\sum_{cyc}a^3\right)-\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)\right)\le$$

$$\le abc\left(3\left(\sum_{cyc}a^3\right)-\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)\right)\overset{?}{\le}$$

$$3\le\left(\left(\sum_{cyc}ab^3\right)-abc\left(\sum_{cyc}a\right)\right)\left(\sum_{cyc}ab\right)$$

$$\Leftrightarrow 3\sum_{cyc}a^2b^4+abc\left(\sum_{cyc}ab^2\right)\underset{(**)}{\overset{?}{\ge}}6a^2b^2c^2+2abc\left(\sum_{cyc}a^2b\right)$$

$$\sum_{cyc}a^2b^4\ge ab^2.bc^2+bc^2.ca^2+ca^2.ab^2=abc\left(\sum_{cyc}a^2b\right)\therefore\text{LHS of }(**)\ge$$

$$\ge 3abc\left(\sum_{cyc}a^2b\right)+abc\left(\sum_{cyc}ab^2\right)\overset{?}{\ge}6a^2b^2c^2+2abc\left(\sum_{cyc}a^2b\right)\Leftrightarrow$$

$$\sum_{cyc}a^2b+\sum_{cyc}ab^2\overset{?}{\ge}6abc\to\text{true}\because\sum_{cyc}a^2b+\sum_{cyc}ab^2\overset{AM-GM}{\ge}6abc$$

$$\therefore\sum_{cyc}\frac{a^2}{b}\ge\lambda.\frac{\sum_{cyc}a^3}{\sum_{cyc}ab}+\left(1-\frac{\lambda}{3}\right)\sum_{cyc}a\ \forall\, a,b,c>0\text{ and }0\le\lambda\le 1,$$

$$''=''\text{ iff }a=b=c\ (\text{QED})$$

1574. If $a,b>0$, $ab=1$, $\lambda>0$ then:

$$\frac{a+\lambda}{b^2}+\frac{(b+\lambda)}{a^2}+\frac{12\lambda}{a+b}\ge 2(4\lambda+1)$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$a+b\overset{AM-GM}{\ge}2\sqrt{ab}\overset{ab=1}{=}2\ or\ t\overset{a+b=t}{\ge}2\ (1)$$

$$\frac{a+\lambda}{b^2}+\frac{(b+\lambda)}{a^2}+\frac{12\lambda}{a+b}=\frac{a^3+b^3+\lambda(a^2+b^2)}{a^2b^2}+\frac{12\lambda}{a+b}=$$
$$=\frac{(a+b)^3-3ab(a+b)+\lambda((a+b)^2-2ab)}{a^2b^2}+\frac{12\lambda}{a+b}=$$
$$\overset{ab=1\ \&\ a+b=t}{=}=\frac{t^3+\lambda t^2-3t-2\lambda}{1}+\frac{12\lambda}{t}=\frac{t^4+\lambda t^3-3t^2-2\lambda t+12\lambda}{t}$$

We need to show:

$$\frac{t^4+\lambda t^3-3t^2-2\lambda t+12\lambda}{t}\ge 2(4\lambda+1)$$
$$t^4+\lambda t^3-3t^2-2\lambda t+12\lambda\ge 8t\lambda+2t$$

$$t^4+\lambda t^3-3t^2-10\lambda t-2t+12\lambda\ge 0$$

$$(t-2)(t^3+\lambda(t^2-3)+\lambda(2t-3)+t)\ge 0\ true\ as\ t\ge 2$$

Equality holds for a=b=1.

1575. If $a,b,c>0$, $abc=1$ then:

$$(a+2b)^{a+2b}.(b+2c)^{b+2c}.(c+2a)^{c+2a}\ge 19683$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$(a+2b)+(b+2c)+(c+2a)=3(a+b+c)\overset{AM\ge GM}{\ge}3\times 3\sqrt[3]{abc}\overset{abc=1}{=}9\ \ (1)$$

Apply weighted $GM\ge HM$:

$$(a+2b)^{a+2b}.(b+2c)^{b+2c}.(c+2a)^{c+2a}\ge$$

$$\geq\left(\frac{(a+2b)+(b+2c)+(c+2a)}{\frac{a+2b}{a+2b}+\frac{b+2c}{b+2c}+\frac{c+2a}{c+2a}}\right)^{a+2b+b+2c+c+2a}\overset{(1)}{\geq}\left(\frac{9}{3}\right)^{9}=3^9=19683$$

Equality holds for $a=b=c=1$.

1576. **If $a,b>0$ then:**

$$\left(\frac{a}{a+1}\right)^b\cdot\left(\frac{b}{b+1}\right)^a\leq\left[1-\frac{4(a+b+1)}{(a+b)^2+4(a+b)+4}\right]^{\frac{a+b}{2}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{a+b}{2}\overset{AM-GM}{\geq}\sqrt{ab}\ or\ (a+b)\geq 2\sqrt{ab}\ or,(a+b)^2\geq 4\,ab\ or\ x^2\overset{a+b=x,ab=y}{\geq}4y\ \ (1)$$

We consider $\frac{a}{a+1}$ *with associated weight* b, and $\frac{b}{b+1}$ *with associated weight* c.

Now we apply $GM\leq AM$:

$$\left(\frac{a}{a+1}\right)^b\cdot\left(\frac{b}{b+1}\right)^a\leq\left(\frac{b.\frac{a}{a+1}+a.\frac{b}{b+1}}{a+b}\right)^{a+b}=$$

$$=\left(\frac{ab\left(\frac{1}{a+1}+\frac{1}{b+1}\right)}{a+b}\right)^{a+b}=\left(\frac{ab(a+b+2)}{(a+b)(a+1)(b+1)}\right)^{a+b}=$$

$$=\left(\frac{ab(a+b+2)}{(a+b)(a+b+ab+1)}\right)^{a+b}\overset{a+b=x,ab=y}{=}\left(\frac{y(x+2)}{x(1+x+y)}\right)^{x}\ (2)$$

$$Now\ we\ need\ to\ show:\left(\frac{a}{a+1}\right)^b\cdot\left(\frac{b}{b+1}\right)^a\leq\left[1-\frac{4(a+b+1)}{(a+b)^2+4(a+b)+4}\right]^{\frac{a+b}{2}}$$

$$\left(\frac{a}{a+1}\right)^b\cdot\left(\frac{b}{b+1}\right)^a\overset{a+b=x,ab=y}{\leq}\left[1-\frac{4(x+1)}{x^2+4x+4}\right]^{\frac{x}{2}}=$$

$$=\left(1-\frac{4(x+1)}{(x+2)^2}\right)^{\frac{x}{2}}=\left(\frac{x^2}{(x+2)^2}\right)^{\frac{x}{2}}=\left(\frac{x}{x+2}\right)^x$$

$$\left(\frac{y(x+2)}{x(1+x+y)}\right)^x\overset{using\ (2)}{\leq}\left(\frac{x}{x+2}\right)^x$$

$$or\ \frac{y(x+2)}{x(1+x+y)} \le \frac{x}{x+2}\ or\ (x+2)y(x+2) \le x^2(1+x+y)$$
$$or\ \ y(x^2+4x+4) \le x^2(1+x+y)\ or\ \ 4y(x+1) \le x^2(1+x)$$

$$or\ (1+x)(x^2-4y) \ge 0\ true\ by\ (1)$$

Equality holds for a=b=1

1577. If $a, b > 0$, $a+b = 2a^2b^2$ then:

$$\frac{1}{a^2} + \frac{1}{b^2} \ge 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$a+b = 2a^2b^2\ or\ (a+b) = 2(ab)^2\ or\ a+b \overset{AM-GM}{\le} 2\left(\frac{(a+b)^2}{4}\right)^2$$

$$a+b \le \frac{(a+b)^4}{8}\ or\ (a+b)^3 \ge 8\ or\ a+b \ge 2\ \ (1)$$

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{a^2+b^2}{a^2b^2} = \frac{2(a^2+b^2)}{2a^2b^2} \overset{CBS\ \&\ a+b=2a^2b^2}{\ge} \frac{(a+b)^2}{a+b} = a+b \overset{(1)}{\ge} 2$$

$$Equality\ holds\ for\ a=b=1.$$

1578. If $a, b, c > 0$ and $a+b+c = 3$, then prove that :

$$\frac{a^3}{3a-ab-ac+2bc} + \frac{b^3}{3b-bc-ba+2ca} + \frac{c^3}{3c-ca-cb+2ab} + 3abc \le 4$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^3}{3a-ab-ac+2bc} + \frac{b^3}{3b-bc-ba+2ca} + \frac{c^3}{3c-ca-cb+2ab}$$
$$+3abc-4 \overset{a+b+c=3}{=} \sum_{cyc} \frac{a^3}{a(a+b+c)-ab-ac+2bc} + 3abc - 4$$

$$= \sum_{cyc} \frac{a(a^2+2bc-2bc)}{a^2+2bc} + 3abc - 4 = \sum_{cyc} a - 2abc.\sum_{cyc} \frac{1}{a^2+2bc} + 3abc - 4$$

$$\overset{\text{Bergstrom}}{\le} 3+3abc-4-2abc.\frac{9}{\sum_{cyc}a^2+2\sum_{cyc}bc}=3abc-1-2abc.\frac{9}{\left(\sum_{cyc}a\right)^2}$$

$$\overset{a+b+c=3}{=}3abc-1-2abc.\frac{9}{9}=abc-1\overset{AM-GM}{\le}\frac{1}{27}.\left(\sum_{cyc}a\right)^3-1\overset{a+b+c=3}{=}$$

$$\frac{1}{27}.27-1=0\Rightarrow\frac{a^3}{3a-ab-ac+2bc}+\frac{b^3}{3b-bc-ba+2ca}+$$

$$\frac{c^3}{3c-ca-cb+2ab}+3abc\le 4\forall\, a,b,c>0\,\middle|\,a+b+c=3,$$

$$''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1579. If $a,b>0$ then:

$$\left(\frac{a+1}{b}\right)^a.\left(\frac{b+1}{b}\right)^b\ge\left[1+\frac{4(a+b+1)}{(a+b)^2}\right]^{\frac{a+b}{2}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\left(\frac{a+1}{b}\right)^a.\left(\frac{b+1}{b}\right)^b\overset{GM-HM}{\ge}\left(\frac{a+b}{\frac{a}{\frac{a+1}{b}}+\frac{b}{\frac{b+1}{a}}}\right)^{a+b}=\left(\frac{a+b}{\frac{ab}{a+1}+\frac{ab}{b+1}}\right)^{a+b}=$$

$$=\left(\frac{a+b}{\frac{ab(a+b+2)}{a+b+ab+1}}\right)^{a+b}$$

$$=\left(\frac{(a+b)(1+a+b+ab)}{ab(a+b+2)}\right)^{a+b}\overset{a+b=x,ab=y}{=}\left(\frac{x(1+x+y)}{y(x+2)}\right)^x\ (1)$$

$$\left[1+\frac{4(a+b+1)}{(a+b)^2}\right]^{\frac{a+b}{2}}\overset{a+b=x,ab=y}{=}\left(1+\frac{4(x+1)}{x^2}\right)^{\frac{x}{2}}=$$

$$=\left(\frac{x^2+4x+4}{x^2}\right)^{\frac{x}{2}}=\left(\frac{(x+2)^2}{x^2}\right)^{\frac{x}{2}}=\left(\frac{x+2}{x}\right)^x\ (2)$$

$$(a+b)\overset{AM-GM}{\ge}2\sqrt{ab}\ or, x\ge 2\sqrt{y}\ (a+b=x,ab=y) or, x^2\ge 4y\ (3)$$

We need to show :

$$\left(\frac{a+1}{b}\right)^a\cdot\left(\frac{b+1}{b}\right)^b\ge\left[1+\frac{4(a+b+1)}{(a+b)^2}\right]^{\frac{a+b}{2}}$$

$$\left(\frac{x(1+x+y)}{y(x+2)}\right)^x\overset{(1)\&(2)}{\ge}\left(\frac{x+2}{x}\right)^x\ or\ \frac{x(1+x+y)}{y(x+2)}\ge\frac{x+2}{x}$$

$$x^2(1+x+y)\ge y(x+2)^2\ or\ x^2+x^3+x^2y\ge y(x^2+4x+4)$$

$$x^3+x^2\ge 4xy+4y\ \ or\ x^2(x+1)\ge 4y(x+1)\ or\ x^2\ge 4y\ true\ by\ \ (3)$$

Equality holds for a=b=1.

1580. **If $a,b,c>0$, $a^2b^2+b^2c^2+c^2a^2=abc$ then:**

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge 27$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$a^2b^2+b^2c^2+c^2a^2=abc\ or,3(abc)^{\frac{4}{3}}\overset{AM-GM}{\le}abc\ \ or,(abc)^{\frac{1}{3}}\le\frac{1}{3}\ (1)$$

$$We\ need\ to\ show:\ \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge 27$$

$$(a^2b^2+b^2c^2+c^2a^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\ge 27(a^2b^2+b^2c^2+c^2a^2)$$

$$(a^2b^2+b^2c^2+c^2a^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\ge 27abc\ (as\ (a^2b^2+b^2c^2+c^2a^2)=abc)$$

$$(a^2b^2+b^2c^2+c^2a^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\overset{C-S}{\ge}(b+c+a)^2\overset{AM-GM}{\ge}9(abc)^{\frac{2}{3}}=$$

$$=27(abc)^{\frac{2}{3}}\cdot\frac{1}{3}\overset{(1)}{\ge}27(abc)^{\frac{2}{3}}.(abc)^{\frac{1}{3}}=27abc$$

$$Equality\ holds\ for\ a=b=c=\frac{1}{3}.$$

1581. **If $a,b,c>0$, $abc\ge 1$ then:**

$$(a^2+b^2)^{a+b}(c^2+b^2)^{c+b}(a^2+c^2)^{a+c}\ge 64$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$(a^2+b^2)^{a+b}(c^2+b^2)^{c+b}(a^2+c^2)^{a+c} \underbrace{\geq}_{A-G}$$

$$\geq (2ab)^{a+b}.(2bc)^{c+b}.(2ac)^{a+c} \underbrace{\geq}_{Weighted\ (GM\geq HM)}$$

$$\geq \left(\frac{2(a+b+c)}{\frac{2ab}{a+b}+\frac{2bc}{b+c}+\frac{2ac}{a+c}}\right)^{2(a+b+c)} = \left(\frac{(a+b+c)}{\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ac}{a+c}}\right)^{2(a+b+c)} \underbrace{\geq}_{A-G}$$

$$\geq \left(\frac{(a+b+c)}{\frac{a+b}{4}+\frac{b+c}{4}+\frac{a+c}{4}}\right)^{2(a+b+c)} = (2)^{2(a+b+c)} \underbrace{\geq}_{A-G} 2^{2.3\sqrt[3]{abc}} \geq 2^6 = 64$$

$$Equality\ \ holds\ if:\ a=b=c=1.$$

1582. If $x, y, z > 0$ and $x+y+z=1$ then prove that :

$$\sqrt[7]{x^3y^2z}+\sqrt[7]{y^3z^2x}+\sqrt[7]{z^3x^2y} \leq \sqrt[7]{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Via Holder}, \sum_{cyc}\sqrt[7]{x^3y^2z} \leq \sqrt[7]{729\left(\sum_{cyc}x^3y^2z\right)} = \sqrt[7]{729xyz\sum_{cyc}x^2y} \overset{CBS}{\leq}$$

$$\sqrt[7]{729xyz.\sqrt{\left(\sum_{cyc}x^2y^2\right)\left(\sum_{cyc}x^2\right)}} \overset{?}{\leq} \sqrt[7]{3} \Leftrightarrow 243xyz.\sqrt{\left(\sum_{cyc}x^2y^2\right)\left(\sum_{cyc}x^2\right)} \overset{?}{\leq} 1$$

$$\overset{x+y+z=1}{=} \left(\sum_{cyc}x\right)^6 \Leftrightarrow \left(\sum_{cyc}x\right)^{12} \underbrace{\overset{?}{\geq}}_{(*)} 59049x^2y^2z^2\left(\sum_{cyc}x^2y^2\right)\left(\sum_{cyc}x^2\right)$$

Assigning $y+z=a, z+x=b, x+y=c \Rightarrow a+b-c=2z>0, b+c-a=2x > 0$ and $c+a-b=2y>0 \Rightarrow a+b>c, b+c>a, c+a>b \Rightarrow a, b, c$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say)

$$\Rightarrow \sum_{cyc}x = s, xyz = r^2s, \sum_{cyc}x^2 = s^2-8Rr-2r^2 \text{ and}$$

$$\sum_{cyc}x^2y^2 = r^2\left((4R+r)^2-2s^2\right) \text{and so}, (*) \Leftrightarrow$$

$$s^{10} \underbrace{\overset{?}{\geq}}_{(**)} 59049r^6\left((4R+r)^2-2s^2\right)\left(s^2-8Rr-2r^2\right) \text{ and } \because P =$$

$$\left(s^2-16Rr+5r^2\right)^5 + r(80R-25r)\left(s^2-16Rr+5r^2\right)^4$$

$$+2r^2(1280R^2-800Rr+125r^2)(s^2-16Rr+5r^2)^3+$$
$$4r^3(1024R^3-9600R^2r+3000Rr^2+2912r^3)(s^2-16Rr+5r^2)^2+$$
$$4r^4\begin{pmatrix}81920R^4-102400R^3r-188196R^2r^2+580490Rr^3\\-368275r^4\end{pmatrix}(s^2-16Rr+5r^2)$$
$$\overset{\text{Gerretsen}}{\geq} 0$$
$$\begin{pmatrix}\because 81920R^4-102400R^3r-188196R^2r^2+580490Rr^3-368275r^4\\=(R-2r)(81920R^3+61440R^2r-653160Rr^2+449858r^3)+531441r^4\\\overset{\text{Euler}}{\geq} 531441r^4>0\end{pmatrix}$$
$$\therefore \text{in order to prove } (**), \text{it suffices to prove : LHS of } (**)\overset{?}{\geq} P \Leftrightarrow$$
$$65536t^5-102400t^4-408392t^3+1101931t^2-941659t+283978\overset{?}{\geq}0$$
$$\left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)\begin{pmatrix}(t-2)(65536t^3+159744t^2-31560t+336715)\\+531441\end{pmatrix}\overset{?}{\geq}0$$
$$\rightarrow \text{true} \because t\overset{\text{Euler}}{\geq}2\Rightarrow(**)\Rightarrow(*)\text{is true} \therefore \sqrt[7]{x^3y^2z}+\sqrt[7]{y^3z^2x}+\sqrt[7]{z^3x^2y}\leq$$
$$\sqrt[7]{3}\ \forall\, x,y,z>0\,\big|\,x+y+z=1,''='' \text{ iff } x=y=z=\frac{1}{3}\ \text{(QED)}$$

1583. **If $a,b,c>0$, $abc=1$ then:**

$$\frac{\sqrt{bc(b^2+c^2)}+b\sqrt{c^2+a^2}}{a\sqrt{b}+c\sqrt{a}}+\frac{\sqrt{ac(a^2+c^2)}+c\sqrt{a^2+b^2}}{b\sqrt{c}+a\sqrt{b}}+\frac{\sqrt{ab(a^2+b^2)}+a\sqrt{b^2+c^2}}{c\sqrt{a}+b\sqrt{c}}\geq 3\sqrt{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

Walter Janous inequaliy: $x,y,z;\ A,B,C>0$ *then*:

$$\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)\geq\sqrt{3(AB+BC+CA)}\ (1)$$

$$\frac{\sqrt{bc(b^2+c^2)}+b\sqrt{c^2+a^2}}{a\sqrt{b}+c\sqrt{a}}\overset{AM-GM}{\geq}\frac{\sqrt{bc.2bc}+b\sqrt{2ca}}{a\sqrt{b}+c\sqrt{a}}=\frac{\sqrt{2}b\sqrt{c}(\sqrt{c}+\sqrt{a})}{a\sqrt{b}+c\sqrt{a}}=$$
$$=\sqrt{2}\frac{b\sqrt{c}}{a\sqrt{b}+c\sqrt{a}}(\sqrt{c}+\sqrt{a})=\sqrt{2}\frac{y}{x+z}(C+A)\ (2)$$

$$(x=a\sqrt{b},y=b\sqrt{c},z=c\sqrt{a};\ A=\sqrt{a},B=\sqrt{b},C=\sqrt{c})$$

$$\frac{\sqrt{bc(b^2+c^2)}+b\sqrt{c^2+a^2}}{a\sqrt{b}+c\sqrt{a}}+\frac{\sqrt{ac(a^2+c^2)}+c\sqrt{a^2+b^2}}{b\sqrt{c}+a\sqrt{b}}+\frac{\sqrt{ab(a^2+b^2)}+a\sqrt{b^2+c^2}}{c\sqrt{a}+b\sqrt{c}}$$

$$=\sum\frac{\sqrt{bc(b^2+c^2)}+b\sqrt{c^2+a^2}}{a\sqrt{b}+c\sqrt{a}}\overset{(2)}{\geq}\sum\sqrt{2}\,\frac{y}{x+z}(C+A)\overset{(1)}{\geq}\sqrt{2}\,\sqrt{3(AB+BC+CA)}=$$

$$=\sqrt{2}\,\sqrt{3\sum\sqrt{ab}}\overset{AM-GM}{\geq}\sqrt{2}\,\sqrt{3\times 3\sqrt[3]{abc}}\overset{abc=1}{=}3\sqrt{2}$$

Equality holds for $a=b=c=1$.

1584. If $x,y>0$ and $(\sqrt{x}+1)(2\sqrt{y}+4)+y\geq 13$ then prove that :

$$\frac{x^4}{y}+\frac{y^3}{x}+y\geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(\sqrt{x}+1)(2\sqrt{y}+4)+y\geq 13\Rightarrow 2\sqrt{xy}+4\sqrt{x}+2\sqrt{y}+y\geq 9$$

$$\Rightarrow(2\sqrt{x}+\sqrt{y})(2+\sqrt{y})\geq 9\Rightarrow 9\leq(2\sqrt{x}+\sqrt{y})(2+\sqrt{y})\overset{CBS}{\leq}$$

$$\sqrt{3}.\sqrt{2x+y}.\sqrt{3}.\sqrt{y+2}\overset{AM-GM}{\leq}\frac{3}{2}(2x+2y+2)\Rightarrow x+y\overset{①}{\geq}2$$

$\boxed{\text{Case 1}}$ $x\leq\frac{1}{4}$ and then : $y\overset{\text{via ①}}{\geq}2-x\geq 2-\frac{1}{4}=\frac{7}{4}$ and so, $\frac{x^4}{y}+\frac{y^3}{x}+y-3$

$$>\frac{y^3}{x}+y-3\geq 4y^3+y-3>4\left(\frac{7}{4}\right)^3+\frac{7}{4}-3>0\therefore\frac{x^4}{y}+\frac{y^3}{x}+y>3$$

$\boxed{\text{Case 2}}$ $\frac{1}{4}<x<2$ and then : $\frac{x^4}{y}+\frac{y^3}{x}-(3-y)=\frac{x^4}{y}+\frac{y^4}{xy}-(3-y)\overset{\text{Bergstrom}}{\geq}$

$$\frac{(x^2+y^2)^2}{y(x+1)}-(3-y)\overset{\substack{\text{AM-GM}\\ \text{and}\\ \text{via ①}}}{\geq}\frac{(x^2+(2-x)^2).2xy}{y(x+1)}-(x+1)=$$

$$\frac{2x(x^2+(2-x)^2)-(x+1)^2}{x+1}=\frac{(x-1)^2(4x-1)}{x+1}\geq 0\because\frac{1}{4}<x\therefore\frac{x^4}{y}+\frac{y^3}{x}+y\geq 3$$

$\boxed{\text{Case 3}}$ $x\geq 2$ and then : $\frac{x^4}{y}+\frac{y^3}{x}-(3-y)=\frac{x^4}{y}+\frac{y^4}{xy}-(3-y)\overset{\text{Bergstrom}}{\geq}$

$$\frac{(x^2+y^2)^2}{y(x+1)}-(3-y)\overset{\substack{\text{AM-GM}\\ \text{and}\\ \text{via ①}}}{\geq}\frac{2xy(x^2+y^2)}{y(x+1)}-(x+1)>\frac{2x(x^2)}{x+1}-(x+1)$$

$$=\frac{2x^3-x^2-2x-1}{x+1}=\frac{(x-2)(2x^2+3x+4)+7}{x+1}>0\because x\geq 2$$

$$\therefore \frac{x^4}{y}+\frac{y^3}{x}+y>3 \text{ and so, combining all cases, } \frac{x^4}{y}+\frac{y^3}{x}+y\geq 3\ \forall$$
$$x,y>0 \mid (\sqrt{x}+1)(2\sqrt{y}+4)+y\geq 13, ''='' \text{ iff } x=y=1 \text{ (QED)}$$

1585. If $a\geq b\geq c>0$ then prove that :

$$\frac{a^3b}{c}+b^2c+c^3\geq\frac{a^2b^2}{c}+bc^2+c^2a$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^3b}{c}+b^2c+c^3\geq\frac{a^2b^2}{c}+bc^2+c^2a$$
$$\Leftrightarrow a^3b+b^2c^2+c^4-a^2b^2-bc^3-c^3a\geq 0$$
$$\Leftrightarrow (b^2c^2-a^2b^2)-(bc^3-a^3b)+(c^4-c^3a)\geq 0$$
$$\Leftrightarrow (c-a)(b^2c+ab^2-bc^2-abc-a^2b+c^3)\geq 0$$
$$\Leftrightarrow (c-a)\Big((c^3-bc^2)+(ab^2-a^2b)+(b^2c-abc)\Big)$$
$$\Leftrightarrow (c-a)\Big(c^2(c-b)+ab(b-a)+bc(b-a)\Big)\geq 0$$
$$\Leftrightarrow (a-c)\Big(c^2(b-c)+ab(a-b)+bc(a-b)\Big)\geq 0\to \text{true} \because a\geq b\geq c>0$$
$$\therefore \frac{a^3b}{c}+b^2c+c^3\geq\frac{a^2b^2}{c}+bc^2+c^2a\ \forall\ a\geq b\geq c>0, ''='' \text{ iff } a=b=c \text{ (QED)}$$

1586. Let x,y,z,m,n be positive numbers such that $xyz\geq 1$. Prove that :

$$\frac{1}{x^{m+n}(y^m+z^n)+1}+\frac{1}{y^{m+n}(z^m+x^n)+1}+\frac{1}{z^{m+n}(x^m+y^n)+1}\leq 1 \text{ and the}$$

inequality sign is reversed for $xyz\leq 1$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $x^m=a, y^m=b, z^m=c, x^n=u, y^n=v, z^n=w$ and then,

$$\text{proposed inequality} \Leftrightarrow \frac{1}{abu+uwa+1}+\frac{1}{bcv+vub+1}+\frac{1}{caw+wvc+1}\leq 1$$
$$\overset{\text{clearing denominators and simplifying}}{\Leftrightarrow} (a^2b^2c^2uvw+abcu^2v^2w^2-2)+$$
$$(abcuvw-1)(abu+acw+auw+bcv+buv+cvw)\geq 0, \text{ which is true}$$
whenever $xyz\geq 1$ ($\because xyz\geq 1\Rightarrow abc\geq 1$ and $uvw\geq 1$) and evidently,
the reverse inequality holds whenever $xyz\leq 1$

$$(\because xyz \le 1 \Rightarrow abc \le 1 \text{ and } uvw \le 1)$$

$$\therefore \frac{1}{x^{m+n}(y^m+z^n)+1} + \frac{1}{y^{m+n}(z^m+x^n)+1} + \frac{1}{z^{m+n}(x^m+y^n)+1} \le 1$$

$$\forall\, x,y,z,m,n > \Big|\, xyz \ge 1 \text{ and } \frac{1}{x^{m+n}(y^m+z^n)+1} + \frac{1}{y^{m+n}(z^m+x^n)+1} +$$

$$\frac{1}{z^{m+n}(x^m+y^n)+1} \ge 1\ \forall\, x,y,z,m,n > \Big|\, xyz \le 1, '' ='' \text{ iff } xyz = 1 \text{ (QED)}$$

1587. If $x,y,z > 0, xy+yz+zx = 1$ and $1 \le \lambda \le 3$ then :

$$\sum_{cyc} x^2 + \lambda xyz \sum_{cyc} x \ge 1 + \frac{\lambda}{3}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} x^2 + \lambda xyz \sum_{cyc} x \ge 1 + \frac{\lambda}{3} \Leftrightarrow \sum_{cyc} x^2 - 1 \ge \frac{\lambda}{3}\left(1 - 3xyz\sum_{cyc} x\right)$$

$$\overset{xy+yz+zx=1}{\Leftrightarrow} \left(\sum_{cyc} xy\right)\left(\sum_{cyc} x^2 - \sum_{cyc} xy\right) \ge \frac{\lambda}{3}\left(\left(\sum_{cyc} xy\right)^2 - 3xyz\sum_{cyc} x\right) \text{ and}$$

$$\because 0 < \frac{1}{3} \le \frac{\lambda}{3} \le 1 \therefore \text{ it suffices to prove : } \left(\sum_{cyc} xy\right)\left(\sum_{cyc} x^2 - \sum_{cyc} xy\right) \ge$$

$$\left(\sum_{cyc} xy\right)^2 - 3xyz\sum_{cyc} x \Leftrightarrow \sum_{cyc}(x^3y + xy^3) \ge 2\sum_{cyc} x^2y^2 \to \text{true via AM} - \text{GM}$$

$$\therefore \sum_{cyc} x^2 + \lambda xyz \sum_{cyc} x \ge 1 + \frac{\lambda}{3}\ \forall\, x,y,z > 0 \,\big|\, xy+yz+zx = 1 \text{ and } 1 \le \lambda \le 3,$$

$$'' ='' \text{ iff } x = y = z = \frac{1}{\sqrt{3}} \text{ (QED)}$$

1588. If $x,y,z > 0, x+y+z = 1$ and $0 \le \lambda \le 3$, then :

$$(\lambda+1)\left(\sum_{cyc} xy\right)^2 \le \frac{2\lambda+3}{9}\left(\sum_{cyc} xy\right) + \lambda xyz$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$(\lambda+1)\left(\sum_{cyc}xy\right)^2 \le \frac{2\lambda+3}{9}\left(\sum_{cyc}xy\right)+\lambda xyz \overset{x+y+z=1}{\Leftrightarrow} (\lambda+1)\left(\sum_{cyc}xy\right)^2$$

$$\le \frac{2\lambda+3}{9}\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)^2+\lambda xyz\sum_{cyc}x$$

$$\Leftrightarrow \lambda\left(9\left(\sum_{cyc}xy\right)^2-2\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)^2-9xyz\sum_{cyc}x\right)\le$$

$$3\left(\sum_{cyc}xy\right)\left(\left(\sum_{cyc}x\right)^2-3\sum_{cyc}xy\right) \text{ and it's trivially true when :}$$

$$9\left(\sum_{cyc}xy\right)^2-2\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)^2-9xyz\sum_{cyc}x<0\ (\because \lambda\ge 0)\ \&\ \text{when :}$$

$$9\left(\sum_{cyc}xy\right)^2-2\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)^2-9xyz\sum_{cyc}x\ge 0, \text{then :}$$

$$\lambda\left(9\left(\sum_{cyc}xy\right)^2-2\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)^2-9xyz\sum_{cyc}x\right)\overset{\lambda\le 3}{\le}$$

$$3\left(9\left(\sum_{cyc}xy\right)^2-2\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)^2-9xyz\sum_{cyc}x\right)$$

$$\overset{?}{\le} 3\left(\sum_{cyc}xy\right)\left(\left(\sum_{cyc}x\right)^2-3\sum_{cyc}xy\right)\Leftrightarrow \sum_{cyc}(x^3y+xy^3)-2\sum_{cyc}x^2y^2\overset{?}{\ge}0$$

$$\Leftrightarrow \sum_{cyc}xy(x-y)^2\overset{?}{\ge}0 \to \text{true} \therefore (\lambda+1)\left(\sum_{cyc}xy\right)^2\le\frac{2\lambda+3}{9}\left(\sum_{cyc}xy\right)+\lambda xyz$$

$$\forall\, x,y,z>0 \mid x+y+z=1 \text{ and } 0\le\lambda\le 3, ''='' \text{ iff } x=y=z=\frac{1}{3}\ \text{(QED)}$$

1589. **If $x, y, z > 0$, then prove that :**

$$\frac{4(x^2+y^2+z^2)}{27(xy+yz+zx)}+\frac{x}{7x+y+z}+\frac{y}{x+7y+z}+\frac{z}{x+y+7z}\ge\frac{13}{27}$$

Proposed by Neculai Stanciu, Titu Zvonaru-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{4(x^2+y^2+z^2)}{27(xy+yz+zx)}+\frac{x}{7x+y+z}+\frac{y}{x+7y+z}+\frac{z}{x+y+7z}\overset{\text{Bergstrom}}{\ge}$$

$$\frac{4\sum_{cyc}x^2}{27\sum_{cyc}xy}+\frac{\left(\sum_{cyc}x\right)^2}{7\sum_{cyc}x^2+2\sum_{cyc}xy}=\frac{4m}{27n}+\frac{m+2n}{7m+2n}\left(m=\sum_{cyc}x^2, n=\sum_{cyc}xy\right)$$

$$=\frac{4m(7m+2n)+27n(m+2n)}{27n(7m+2n)}\overset{?}{\ge}\frac{13}{27}\Leftrightarrow 28(m-n)^2\overset{?}{\ge}0\rightarrow \text{true}$$

$$\therefore\frac{4(x^2+y^2+z^2)}{27(xy+yz+zx)}+\frac{x}{7x+y+z}+\frac{y}{x+7y+z}+\frac{z}{x+y+7z}\ge\frac{13}{27}$$

$$\forall x,y,z>0,''='' \text{ iff } x=y=z\ (\text{QED})$$

1590. **If $ab(1+a)(1+b)+bc(1+b)(1+c)+ca(1+c)(1+a)\ge 12$,**

then prove that : $a^2+b^2+c^2\ge 3$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$12\le\sum_{cyc}\left(ab(1+a)(1+b)\right)=\sum_{cyc}ab+\sum_{cyc}a^2b^2+\sum_{cyc}a^2b+\sum_{cyc}ab^2$$

$$\overset{\text{CBS}}{\le}\sum_{cyc}a^2+\frac{1}{3}\left(\sum_{cyc}a^2\right)^2+2\sqrt{\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^2b^2\right)}$$

$$\le\sum_{cyc}a^2+\frac{1}{3}\left(\sum_{cyc}a^2\right)^2+2\sqrt{\left(\sum_{cyc}a^2\right).\frac{1}{3}\left(\sum_{cyc}a^2\right)^2}=3t^2+3t^4+6t^3$$

$$\left(t=\sqrt{\frac{\sum_{cyc}a^2}{3}}\right)\Rightarrow t^2(t+1)^2\ge 4\Rightarrow t^2+t-2\ge 0\Rightarrow (t-1)(t+2)\ge 0$$

$$\Rightarrow \mathbf{t} \geq 1 \left(\because \mathbf{t} = \sqrt{\frac{\sum_{cyc} a^2}{3}} \geq 0 \therefore\right) \Rightarrow \sqrt{\frac{\sum_{cyc} a^2}{3}} \geq 1 \Rightarrow a^2 + b^2 + c^2 \geq 3 \text{ whenever}$$

$$ab(1+a)(1+b) + bc(1+b)(1+c) + ca(1+c)(1+a) \geq 12,$$
$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

1591. **If $x, y, z > 0$ then:**

$$\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)^2 + \frac{6(xy + yz + zx)}{x^2 + y^2 + z^2} \geq 15$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)^2 = \left(\frac{x^2}{xy} + \frac{y^2}{yz} + \frac{z^2}{zx}\right)^2 \overset{CBS}{\geq} \left(\frac{(x+y+z)^2}{xy + yz + zx}\right)^2 =$$

$$= \left(\frac{x^2 + y^2 + z^2 + 2(xy + yz + zx)}{xy + yz + zx}\right)^2 =$$

$$= \left(\frac{x^2 + y^2 + z^2}{xy + yz + zx} + 2\right)^2 \overset{m = \frac{x^2+y^2+z^2}{xy+yz+zx} \geq 1}{=} (m+2)^2 \ (1)$$

$$\textit{We need to show}:$$

$$\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)^2 + \frac{6(xy + yz + zx)}{x^2 + y^2 + z^2} \geq 15$$

$$(m+2)^2 + \frac{6}{m} \geq 15 \left(\textit{using } (1) \textit{and } m = \frac{x^2 + y^2 + z^2}{xy + yz + zx} \geq 1\right)$$

$$m^2 + 4m + 4 + \frac{6}{m} = 15 \ or, m^3 + 4m^2 - 11m + 6 \geq 0$$

$$(m-1)^2(m+6) \geq 0 \ \ \textit{true as } m \geq 1$$

$$\textit{Equality holds for } \ a = b = c.$$

1592. **If $x, y, z > 0$, $xyz(x + y + z) = 3$ then:**

$$243(x^2 + y^2 + z^2) \leq (x^5 - 2x + 4)^2(y^5 - 2y + 4)^2(z^5 - 2z + 4)^2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\forall \textit{ real number } x, \textit{we have} (\ x^2 - 1) \textit{and } (x^3 - 1)$$
$$\textit{have the same signs and because of this}$$

$$x^5 - x^3 - x^2 + 1 = (x^2 - 1)(x^3 - 1) \geq 0 \Rightarrow x^5 - x^2 + 3 \geq x^3 + 2 \ (1)$$

$$x^5 - 2x + 4 \overset{AM-GM}{\geq} x^5 - (x^2 + 1) + 4 = x^5 - x^2 + 3 \overset{(1)}{\geq} x^3 + 2$$

$$Similarly: y^5 - 2y + 4 \geq y^3 + 2 \ and \ z^3 - 2z + 4 \geq z^3 + 2$$

$$3 = xyz(x + y + z) = xy.yz + yz.zx + zx.xy \leq \frac{(xy + yz + zx)^2}{3}$$
$$\Rightarrow (xy + yz + zx)^2 \geq 9 \ or \ (xy + yz + zx) \geq 3 \ (2)$$

$$LHS = 243(x^2 + y^2 + z^2) = 243\left((x + y + z)^2 - 2(xy + yz + zx)\right) \overset{(2)}{\leq}$$
$$\leq 243((x + y + z)^2 - 2 \times 3) = 3^5(x + y + z)^2 - 2.3^6 \ (3)$$

$$(x^5 - 2x + 4)(y^5 - 2y + 4)(z^3 - 2z + 4) \overset{(1)}{\geq} (x^3 + 2)(y^3 + 2)(z^3 + 2)$$
$$= (x^3 + 1 + 1)(1 + y^3 + 1)(1 + 1 + z^3) \overset{Holder}{\geq} (x + y + z)^3 \ (4)$$

$$RHS = (x^5 - 2x + 4)^2(y^5 - 2y + 4)^2(z^5 - 2z + 4)^2 \overset{(4)}{\geq} (x + y + z)^6$$

We need to show:

$$(x + y + z)^6 \overset{(3)\&(4)}{\geq} 3^5(x + y + z)^2 - 2.3^6$$

$or, (x + y + z)^6 + 2.3^6 \geq 3^5(x + y + z)^2$ *this is true because*

$$(x + y + z)^6 + 2.3^6 = (x + y + z)^6 +.3^6 + 3^6 \overset{AM-GM}{\geq}$$
$$\geq 3(x + y + z)^2.3^2.3^2 = 243(x + y + z)^2$$

Equality holds for $x = y = z = 1$.

1593. If $x, y, z > 0, xy + yz + zx = 3$ then:

$$\sqrt{7x^2 + y + \frac{1}{z}} + \sqrt{7y^2 + z + \frac{1}{x}} + \sqrt{7z^2 + x + \frac{1}{y}} \geq 3\sqrt{3(x + y + z)}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$x + y + z = \sqrt{(x + y + z)^2} \geq \sqrt{3(xy + yz + zx)} \overset{xy+yz+zx=3}{=} \sqrt{9} = 3 \ (1)$$

Applying generalised Minkowski's inequality, we get:

$$\sqrt{7x^2+y+\frac{1}{z}}+\sqrt{7y^2+z+\frac{1}{x}}+\sqrt{7z^2+x+\frac{1}{y}}\geq$$

$$\geq\sqrt{\left(\sqrt{7}(x+y+z)\right)^2+\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2+\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{z}}\right)^2}\geq$$

$$\overset{AM-GM}{\geq}\sqrt{7(x+y+z)^2+2\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{z}}\right)}\overset{C-S}{\geq}$$

$$\geq\sqrt{7(x+y+z)^2+2\times 9}=\sqrt{7(x+y+z)^2+18}\quad(2)$$

We need to show:

$$\sqrt{7x^2+y+\frac{1}{z}}+\sqrt{7y^2+z+\frac{1}{x}}+\sqrt{7z^2+x+\frac{1}{y}}\geq 3\sqrt{3(x+y+z)}$$

$$\Rightarrow\sqrt{7(x+y+z)^2+18}\geq 3\sqrt{3(x+y+z)}$$

$$\Rightarrow 7(x+y+z)^2+18\geq 27(x+y+z)$$

$$\Rightarrow 7m^2-27m+18\overset{m=x+y+z\geq 3\ by\ (1)}{\geq}0\Rightarrow(7m-6)(m-3)\geq 0\ true\ as\ m\geq 3$$

Equality holds for $x=y=z=1$.

1594. If $a,b,c>0$, then prove that:

$$\sum_{cyc}\frac{\frac{b^4}{a^4}(c^3+a^3)+\frac{c^4}{a^4}(b^3+c^3)}{b^4+c^4}\geq\frac{6}{\sqrt[3]{abc}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\forall\, A',B',C',x',y',z'>0,$$

$$\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')\geq\sqrt{3\sum_{cyc}A'B'}$$

$$(\text{via Walter Janous}) \to ① \text{ and now}, \sum_{cyc} \frac{\frac{b^4}{a^4}(c^3+a^3)+\frac{c^4}{a^4}(b^3+c^3)}{b^4+c^4} =$$

$$\sum_{cyc} \frac{\frac{1}{a^4}\left(\frac{b^3+c^3}{b^4}+\frac{c^3+a^3}{c^4}\right)}{\frac{1}{b^4}+\frac{1}{c^4}} = \frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')$$

$$\left(x'=\frac{1}{a^4}, y'=\frac{1}{b^4}, z'=\frac{1}{c^4}, A'=\frac{a^3+b^3}{a^4}, B'=\frac{b^3+c^3}{b^4}, C'=\frac{c^3+a^3}{c^4}\right) \overset{\text{via ①}}{\geq}$$

$$\sqrt{3\sum_{cyc}\left(\frac{a^3+b^3}{a^4}\cdot\frac{b^3+c^3}{b^4}\right)} \overset{\substack{\text{AM-GM}\\ \text{and}\\ \text{Cesaro}}}{\geq} 3.\sqrt[6]{\left(\frac{8a^3b^3c^3}{a^4b^4c^4}\right)^2} = \frac{6}{\sqrt[3]{abc}} \ \forall a,b,c>0,$$

$$''='' \text{ iff } a=b=c$$

Solution 2 by Mirsadix Muzefferov-Azerbaijan

Let's transform the left side of the expression according to the Walter Janous inequality

$$\frac{\frac{b^4}{a^4}(c^3+a^3)+\frac{c^4}{a^4}(b^3+c^3)}{b^4+c^4} = \frac{\frac{b^4c^3}{a^4}+\frac{b^4}{a}+\frac{c^4b^3}{a^4}+\frac{c^7}{a^4}}{b^4+c^4} =$$

$$= \frac{\frac{1}{a^4}(b^4c^3+a^3b^4+c^4b^3+c^7)}{b^4+c^4} = \frac{\frac{1}{a^4}\cdot\frac{b^4c^3+a^3b^4+c^4b^3+c^7}{b^4.c^4}}{\frac{b^4+c^4}{b^4.c^4}} =$$

$$= \frac{\frac{1}{a^4}\left(\frac{a^3+c^3}{c^4}+\frac{b^3+c^3}{b^4}\right)}{\frac{1}{b^4}+\frac{1}{c^4}} = \frac{\frac{1}{a^4}}{\frac{1}{b^4}+\frac{1}{c^4}}\left(\frac{a^3+c^3}{c^4}+\frac{b^3+c^3}{b^4}\right) = \frac{x}{y+z}(B+C)$$

Walter Janous inequality $x,y,z,A,B,C>0$

$$(*)\quad \frac{x}{y+z}(B+C)+\frac{y}{x+z}(A+C)+\frac{z}{y+x}(B+A) \geq \sqrt{3(AB+BC+CA)}$$

$$\textit{Analogously}: \frac{\frac{1}{b^4}}{\frac{1}{a^4}+\frac{1}{c^4}}\left(\frac{a^3+b^3}{a^4}+\frac{a^3+c^3}{c^4}\right) = \frac{y}{x+z}(A+C)$$

$$\frac{\frac{1}{c^4}}{\frac{1}{a^4}+\frac{1}{b^4}}\left(\frac{c^3+b^3}{b^4}+\frac{a^3+b^3}{a^4}\right) = \frac{z}{x+y}(A+B)$$

$$\textit{Here}: \left(x=\frac{1}{a^4}, y=\frac{1}{b^4}, z=\frac{1}{c^4}, A=\frac{a^3+b^3}{a^4}, B=\frac{c^3+b^3}{b^4}, C=\frac{a^3+c^3}{c^4}\right)$$

$$LHS \overset{Walter\ Janous}{\geq} \sqrt{3(AB+BC+CA)} \overset{AM-GM}{\geq} \sqrt{3.3\sqrt[3]{A^2.B^2.C^2}} =$$

$$= 3.\sqrt[3]{ABC} = 3\sqrt[3]{\frac{a^3+b^3}{a^4}.\frac{c^3+b^3}{b^4}.\frac{a^3+c^3}{c^4}} \overset{AM-GM}{\geq}$$

$$\geq 3\sqrt[3]{\frac{2\sqrt{a^3b^3}}{a^4}.\frac{2\sqrt{b^3c^3}}{b^4}.\frac{2\sqrt{c^3a^3}}{c^4}} = 6\sqrt[3]{\frac{a^3b^3c^3}{a^4b^4c^4}} = \frac{6}{\sqrt[3]{abc}}$$

1595. If $a,b,c > 0$ then:

$$\frac{\sqrt{a+b}}{a^2+b^2}\left(\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)+\frac{\sqrt{b+c}}{b^2+c^2}\left(\frac{1}{\sqrt{c}}+\frac{1}{\sqrt{a}}\right)+\frac{\sqrt{c+a}}{c^2+a^2}\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}\right) \geq \frac{9\sqrt{2}}{a^2+b^2+c^2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\frac{\sqrt{a+b}}{a^2+b^2}\left(\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right) = \frac{\sqrt{a+b}}{a^2+b^2}\left(\frac{1^{\frac{3}{2}}}{\sqrt{b}}+\frac{1^{\frac{3}{2}}}{\sqrt{c}}\right) \overset{Radon}{\geq} \frac{\sqrt{a+b}}{a^2+b^2}\frac{2\sqrt{2}}{\sqrt{b+c}} = \frac{2\sqrt{2}}{a^2+b^2}\sqrt{\frac{a+b}{b+c}} \quad (1)$$

$$\frac{\sqrt{a+b}}{a^2+b^2}\left(\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)+\frac{\sqrt{b+c}}{b^2+c^2}\left(\frac{1}{\sqrt{c}}+\frac{1}{\sqrt{a}}\right)+\frac{\sqrt{c+a}}{c^2+a^2}\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}\right) =$$

$$= \sum \frac{\sqrt{a+b}}{a^2+b^2}\left(\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right) \overset{(1)}{\geq}$$

$$\geq \sum\left(\frac{2\sqrt{2}}{a^2+b^2}\sqrt{\frac{a+b}{b+c}}\right) \overset{AM-GM}{\geq} 6\sqrt{2}\sqrt[3]{\frac{1}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}} \geq$$

$$\overset{AM-GM}{\geq} 6\sqrt{2}\frac{1}{\left(\frac{2(a^2+b^2+c^2)}{3}\right)} = \frac{9\sqrt{2}}{a^2+b^2+c^2}$$

Equality holds for $a=b=c$.

1596.

If $x,y,z > 0$ and $xy+yz+zx+2xyz = 1$ then prove that:

$$\sqrt{1-x^2}+\sqrt{1-y^2}+\sqrt{1-z^2} \leq \frac{3\sqrt{3}}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$1=2xyz+\sum_{cyc}xy\overset{AM-GM}{\geq}2t^3+3t^2\left(t=\sqrt[3]{xyz}\right)\Rightarrow(2t-1)(t+1)^2\leq 0$$

$$\Rightarrow t\leq\frac{1}{2}\to (1)\text{ and }\sum_{cyc}\sqrt{1-x^2}\overset{CBS}{\leq}\sqrt{3\left(3-\sum_{cyc}x^2\right)}\leq\sqrt{3\left(3-\sum_{cyc}xy\right)}$$

$$\overset{xy+yz+zx+2xyz=1}{=}\sqrt{3(3-1+2xyz)}\overset{t\leq\frac{1}{2}}{\leq}\sqrt{3\left(2+\frac{2}{8}\right)}=\frac{3\sqrt{3}}{2}$$

$$\therefore\sqrt{1-x^2}+\sqrt{1-y^2}+\sqrt{1-z^2}\leq\frac{3\sqrt{3}}{2}\ \forall\, x,y,z>0\,|\,xy+yz+zx+2xyz=1,$$

$$''='' \text{ iff } x=y=z=\frac{1}{2}\ (QED)$$

1597. **Let $a,b,c,m,n\geq 0$ and $abc=1$. Prove that**

$$\frac{1}{a^{2m+n}+a^{2n+m}+1}+\frac{1}{b^{2m+n}+b^{2n+m}+1}+\frac{1}{c^{2m+n}+c^{2n+m}+1}\geq 1$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $a^{2m+n}+a^{2n+m}=a^{2(m+n)}+a^{m+n}-a^{m+n}(a^m-1)(a^n-1)$
$\leq a^{2(m+n)}+a^{m+n}$,

because a^m-1, a^n-1 have the same sign.

And since $a^{m+n}.b^{m+n}.c^{m+n}=1$, then $\exists x,y,z>0$ such that

$$a^{m+n}=\frac{yz}{x^2},b^{m+n}=\frac{zx}{y^2},c^{m+n}=\frac{xy}{z^2}.$$

Therefore

$$\sum_{cyc}\frac{1}{a^{2m+n}+a^{2n+m}+1}\geq\sum_{cyc}\frac{1}{a^{2(m+n)}+a^{m+n}+1}=\sum_{cyc}\frac{x^4}{(yz)^2+x^2yz+x^4}$$

$$\overset{AM-GM}{\geq}\sum_{cyc}\frac{x^4}{(yz)^2+\frac{(xy)^2+(zx)^2}{2}+x^4}\overset{CBS}{\geq}\frac{\left(\sum_{cyc}x^2\right)^2}{\sum_{cyc}\left[(yz)^2+\frac{(xy)^2+(zx)^2}{2}+x^4\right]}=1.$$

So the proof is complete. Equality holds iff $a=b=c=1$.

1598. If $a,b>0$ and $a^4+b^4=ab(a+b)$ then:

$$a^2+b^2\le 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$a^4+b^4=\frac{a^4}{1^3}+\frac{b^4}{1^3}\overset{Radon}{\ge}\frac{(a+b)^4}{8}\ (1)$$

$$ab(a+b)\overset{AM-GM}{\le}\frac{(a+b)^2}{4}(a+b)=\frac{(a+b)^3}{4}\ (2)$$

$$a^4+b^4=ab(a+b)\ .\ Using\ (1)\&(2) we\ get:$$
$$\frac{(a+b)^4}{8}\le\frac{(a+b)^3}{4}\ or,(a+b)\le 2\ \ (3)$$
$$We\ need\ to\ show:$$
$$a^2+b^2\le 2\ \ or\ (a+b)^2-2ab\le 2$$

$$(a+b)^2-\frac{2(a^4+b^4)}{a+b}\overset{a^4+b^4=ab(a+b)}{\le}2$$
$$(a+b)^2-\frac{2(a+b)^3}{8}\overset{(1)}{\le}2\ or\ \ 4(a+b)^2-(a+b)^3\le 8$$
$$x^3-4x^2+8\overset{x=a+b}{\ge}0\ or\ (x-2)(x^2-2x-4)\ge 0$$

$$(x-2)(x(x-2)-4)\ge 0\ true\ since\ by\ (3)\ x=a+b\le 2$$

$$so\ (x-2)<0\ and\ (x(x-2)-4)<0\ and (x-2)(x(x-2)-4)\ge 0$$

Equality holds for a=b=1.

1599. If $a+b+c+2=abc$ then:

$$\frac{1}{\sqrt{7+a}}+\frac{1}{\sqrt{7+b}}+\frac{1}{\sqrt{7+c}}\le 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$If\ \ a+b+c+2=abc\ then\ \exists\ x,y,z>0\ such\ that$$

$$a=\frac{x+y}{z},b=\frac{y+z}{x},c=\frac{x+z}{y},$$

$$We\ take\ f(a)=\frac{6a}{6a+s}\ where\ a>0\ ,f''(a)=-\frac{72}{(6a+s)^3}<0,$$
$$so\ f\ is\ concave\ for\ a>0. Using\ Jensen\ inequality\ we\ get:$$

$$\sum\frac{6z}{6z+s}s=f(z)+f(y)+f(x)\geq 3f\left(\frac{x+y+z}{3}\right)\overset{s=x+y+z}{=}3f\left(\frac{s}{3}\right)=$$
$$=3\frac{6\times\frac{s}{3}}{6\times\frac{s}{3}+s}s=2\ (1)$$

$$\frac{1}{\sqrt{7+a}}+\frac{1}{\sqrt{7+b}}+\frac{1}{\sqrt{7+c}}=$$
$$=\sum\frac{1}{\sqrt{7+a}}=\sum\frac{1}{\sqrt{7+\frac{x+y}{z}}}=\sum\sqrt{\frac{z}{6z+(x+y+z)}}\overset{s=x+y+z}{=}$$
$$=\sum\sqrt{\frac{z}{6z+s}}\overset{CBS}{\leq}\sqrt{3\left(\sum\frac{z}{6z+s}\right)}=\sqrt{\frac{3}{6}\left(\sum\frac{6z}{6z+s}\right)}=$$
$$=\sqrt{\frac{1}{2}\left(\sum\frac{z}{6z+s}\right)}\overset{(1)}{\leq}\sqrt{\frac{1}{2}\times 2}=1$$

Equality holds for a=b=c=2.

1600. **If $a,b,c>0$ and $a+b+c+2=abc$, then prove that :**

$$\sqrt{a}+\sqrt{b}+\sqrt{c}\leq\frac{3}{2}\sqrt{abc}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } a=x+1,b=y+1,c=z+1 \text{ and then :}$$
$$x+y+z+5=(x+1)(y+1)(z+1)\Rightarrow xyz+xy+yz+zx=4$$
$$\Rightarrow\sum_{cyc}\big((2+y)(2+z)\big)-(2+x)(2+y)(2+z)=4-(xy+yz+zx+xyz)=0$$
$$\therefore\sum_{cyc}\frac{1}{2+x}=1\rightarrow(m)\ (\because 2+x=a+1>1>0 \text{ and analogs) and setting :}$$

$$\frac{1}{2+x}=\frac{1}{2}-\alpha\Rightarrow x=\frac{2\alpha}{\frac{1}{2}-\alpha}\to(1);\ \text{similarly, setting}:\frac{1}{2+y}=\frac{1}{2}-\beta\ \text{and}$$

$$\frac{1}{2+c}=\frac{1}{2}-\gamma\Rightarrow 1\overset{\text{via (m)}}{=}\frac{1}{2}-\alpha+\frac{1}{2}-\beta+\frac{1}{2}-\gamma\Rightarrow\alpha+\beta+\gamma=\frac{1}{2}\to(\text{i})$$

We note that : $\beta+\gamma=\frac{1}{2}-\alpha=\frac{1}{2+x}>0$ and similarly, $\gamma+\alpha>0$ and $\alpha+\beta>0$

$$\therefore(1)\ \text{and}\ (\text{i})\Rightarrow x=\frac{2\alpha}{\beta+\gamma}\ \text{and analogously}, y=\frac{2\beta}{\gamma+\alpha}\ \text{and}\ z=\frac{2\gamma}{\alpha+\beta}$$

and via such substitutions, $\sqrt{a}+\sqrt{b}+\sqrt{c}\le\frac{3}{2}.\sqrt{abc}$

$$\Leftrightarrow\sqrt{\frac{2\alpha}{\beta+\gamma}+1}+\sqrt{\frac{2\beta}{\gamma+\alpha}+1}+\sqrt{\frac{2\gamma}{\alpha+\beta}+1}\le$$

$$\frac{3}{2}.\sqrt{\left(\frac{2\alpha}{\beta+\gamma}+1\right)\left(\frac{2\beta}{\gamma+\alpha}+1\right)\left(\frac{2\gamma}{\alpha+\beta}+1\right)}$$

$$\Leftrightarrow\sqrt{\frac{Y+Z}{X}}+\sqrt{\frac{Z+X}{Y}}+\sqrt{\frac{X+Y}{Z}}\overset{(*)}{\le}\frac{3}{2}.\sqrt{\frac{(Y+Z)(Z+X)(X+Y)}{XYZ}}$$

$$(X=\beta+\gamma>0;Y=\gamma+\alpha>0;Z=\alpha+\beta>0)$$

$$\text{Now,}\sum_{cyc}\sqrt{\frac{Y+Z}{X}}\overset{CBS}{\le}\sqrt{\sum_{cyc}(Y+Z)}.\sqrt{\sum_{cyc}\frac{1}{X}}=\frac{3}{2}.\sqrt{\frac{8(\sum_{cyc}X)(\sum_{cyc}XY)}{9XYZ}}$$

$$=\frac{3}{2}.\sqrt{\frac{9(\sum_{cyc}X)(\sum_{cyc}XY)-(\sum_{cyc}X)(\sum_{cyc}XY)}{9XYZ}}\overset{AM-GM}{\le}\frac{3}{2}.\sqrt{\frac{9(\sum_{cyc}X)(\sum_{cyc}XY)-9XYZ}{9XYZ}}$$

$$=\frac{3}{2}.\sqrt{\frac{(Y+Z)(Z+X)(X+Y)}{XYZ}}\Rightarrow(*)\ \text{is true}\Rightarrow\sqrt{a}+\sqrt{b}+\sqrt{c}\le\frac{3}{2}.\sqrt{abc}$$

$$\forall a,b,c>0\,|\,a+b+c+2=abc,''=''\ \text{iff}\ x=y=z=1$$

$$\Rightarrow''=''\ \text{iff}\ a=b=c=2\ (\text{QED})$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru