Inequalities with Nesbitt lacunary sums

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In this paper, the notions of Nesbitt lacunary sums are introduced and studied, for which several inequalities are obtained. Among others, a new proof and a refinement of Nesbitt's inequality are obtained. Also, specific inequalities are obtained through convexity and monotonicity considerations.

Keywords : ***Nesbitt's inequality , Nesbitt lacunary sum , convex / concave function , refinement***

2020 ***Mathematics Subject Classification*** **: 26D15**

Is well known and intensively studied, the following inequality ,

▻ ***Nesbitt's inequality,*** [9] :

If $a , b , c > 0$, then , $$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2} \quad , \qquad (1)$$

having equality iff $a=b=c$.

Over time, this inequality has known numerous demonstrations and several extensions and/or generalizations, and there is already a considerable dedicated literature. And from our side it has also been the subject of some works, including [3].

The expression on the left side of inequality (1) we will call the *Nesbitt sum* :

$$\mathcal{N}(a,b,c):=\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \ . \qquad (2)$$

When one of the component fractions is missing from the *Nesbitt sum* , we will call the expression obtained a *lacunary Nesbitt sum*. Hence ,

$$\frac{a}{b+c}+\frac{b}{c+a} \quad , \quad \frac{b}{c+a}+\frac{c}{a+b} \quad , \quad \frac{c}{a+b}+\frac{a}{b+c} \quad , \qquad (3)$$

are *Nesbitt lacunary sums* .

A result that was also used in solving an inequality in [4] is given by the following ,

1. ***Lemma*** If $a , b , c \in \mathbb{R}_{>0}$, then ,

$$\frac{a}{b+c}+\frac{b}{c+a}\geq\frac{a+b}{\sqrt{(a+c)\cdot(b+c)}} \ . \qquad (4)$$

Proof

Using the *QM-AM* and *AM-GM inequalities* , we have successively :

$$\frac{a}{b+c}+\frac{b}{c+a}=\frac{a^2+b^2+c(a+b)}{(b+c)(c+a)}\overset{\text{QM-AM}}{\geq}\frac{\frac{1}{2}\cdot(a+b)^2+c(a+b)}{(b+c)(c+a)}=$$

$$=\frac{(a+b)\cdot(a+b+2c)}{2(b+c)(c+a)}=\frac{(a+b)\cdot\left[(a+c)+(b+c)\right]}{2(b+c)(c+a)}\overset{\text{AM-GM}}{\geq}$$

$$\overset{\text{AM-GM}}{\geq}\frac{(a+b)\cdot\sqrt{(a+c)\cdot(b+c)}}{(b+c)(c+a)}=\frac{a+b}{\sqrt{(a+c)\cdot(b+c)}}\ .$$

Equality occurs (under the conditions of applying the means inequalities in the proof) , if $a=b$ and c arbitrary in $\mathbb{R}_{>0}$.

There are other possibilities for proving the inequality in *Lemma* .

2. ***Proposition*** (inequality for the product of *Nesbitt lacunary sums*) , [4]

If $a,b,c\in\mathbb{R}_{>0}$, then the following inequality occurs ,

$$\left(\frac{a}{b+c}+\frac{b}{c+a}\right)\cdot\left(\frac{b}{c+a}+\frac{c}{a+b}\right)\cdot\left(\frac{c}{a+b}+\frac{a}{b+c}\right)\geq 1\ . \qquad (5)$$

(The product of the three *Nesbitt lacunary sums* is superunitary)

Proof

Using the inequality in *Lemma* 1 and its two other analogues , we immediately have :

$$\prod_{cycl}\left(\frac{a}{b+c}+\frac{b}{c+a}\right)\overset{\text{Lemma 1}}{\geq}\prod_{cycl}\frac{a+b}{\sqrt{(a+c)\cdot(b+c)}}=\frac{\prod_{cycl}(a+b)}{\sqrt{\prod_{cycl}(a+c)\cdot(b+c)}}=1\ .$$

Equality occurs iff $a=b=c$.

If we try to sum of the three *Nesbitt lacunary sums* , we are led to the following ,

3. ***Proposition*** **(A rafinement of *Nesbitt's inequality*)** , [5]

If $a,b,c\in\mathbb{R}_{>0}$, then occur the inequalities ,

$$\sum_{cycl}\frac{a}{b+c}\geq\frac{1}{2}\cdot\sum_{cycl}\frac{a+b}{\sqrt{(b+c)(c+a)}}\geq\frac{3}{2}\ . \qquad (6)$$

Proof

With the help of the *Lemma* 1 and then with the *AM-GM inequality* , we have :

$$\sum_{cyc}\frac{a}{b+c}=\frac{1}{2}\cdot\sum_{cyc}\left(\frac{a}{b+c}+\frac{b}{c+a}\right)\overset{(Lemma\,1)}{\geq}\frac{1}{2}\cdot\sum_{cyc}\frac{a+b}{\sqrt{(b+c)(c+a)}}\overset{(AM-GM)}{\geq}$$

$$\overset{(AM-GM)}{\geq}\frac{1}{2}\cdot 3\cdot\sqrt[3]{\prod_{cyc}\frac{a+b}{\sqrt{(b+c)(c+a)}}}=\frac{3}{2}\ .$$

Equality occurs iff $a=b=c$.

4. *Remark*

The above proof can also be considered a novel proof of *Nesbitt's inequality* , thus adding to the dozens of proofs for this famous inequality.

We will establish another inequality in which *Nesbitt lacunary sums* appear, but not before stating and proving the following result - important in itself.

5. *Lemma* , [6]

If $a,b,c\in\mathbb{R}_{>0}$, then holds the following inequality ,

$$\frac{a}{b+c}\cdot\frac{b}{c+a}+\frac{b}{c+a}\cdot\frac{c}{a+b}+\frac{c}{a+b}\cdot\frac{a}{b+c}\geq\frac{3}{4}\ . \qquad (7)$$

Proof

With the substitutions : $a=\frac{1}{x}$, $b=\frac{1}{y}$, $c=\frac{1}{z}$, the inequality becomes ,

$$\frac{z^2}{(x+z)(y+z)}+\frac{x^2}{(y+x)(z+x)}+\frac{y^2}{(z+y)(x+y)}\geq\frac{3}{4}\ .$$

Using *Bergström's inequality* (B) for the left side of this inequality , and then with the known inequality , $xy+yz+zx\leq\frac{1}{3}(x+y+z)^2$. **(8)**

we successively obtain :

$$\frac{x^2}{(y+x)(z+x)}+\frac{y^2}{(z+y)(x+y)}+\frac{z^2}{(x+z)(y+z)}\overset{(B)}{\geq}\frac{(x+y+z)^2}{x^2+y^2+z^2+3(xy+xy+xy)}=$$

$$=\frac{(x+y+z)^2}{(x+y+z)^2+(xy+xy+xy)}\overset{(8)}{\geq}\frac{(x+y+z)^2}{(x+y+z)^2+\frac{1}{3}\cdot(x+y+z)^2}=\frac{3}{4}\ .$$

The equality holds for $x=y=z\ \Leftrightarrow\ a=b=c$.

6. *Proposition*

If $a,b,c\in\mathbb{R}_{>0}$, then holds the following inequality ,

$$\left(\frac{a}{b+c}+\frac{b}{c+a}\right)\cdot\frac{c}{a+b}+\left(\frac{b}{c+a}+\frac{c}{a+b}\right)\cdot\frac{a}{b+c}+\left(\frac{c}{a+b}+\frac{a}{b+c}\right)\cdot\frac{b}{c+a}\geq\frac{3}{2} \qquad (9)$$

Proof

We have immediately by *Lemma* 5 :

$$\left(\frac{a}{b+c}+\frac{b}{c+a}\right)\cdot\frac{c}{a+b}+\left(\frac{b}{c+a}+\frac{c}{a+b}\right)\cdot\frac{a}{b+c}+\left(\frac{c}{a+b}+\frac{a}{b+c}\right)\cdot\frac{b}{c+a}=$$

$$=2\cdot\left(\frac{a}{b+c}\cdot\frac{b}{c+a}+\frac{b}{c+a}\cdot\frac{c}{a+b}+\frac{c}{a+b}\cdot\frac{a}{b+c}\right)\overset{(Lemma\ 5)}{\geq}2\cdot\frac{3}{4}=\frac{3}{2}\ .$$

Equality occurs iff $a=b=c$.

Trying to obtain an inequality for power sums of *Nesbitt lacunary sums* ,I found the following,

7. *Proposition* , [7]

If $a,b,c\in\mathbb{R}_{>0}$ and $p\in\mathbb{R}_{\geq 0}$, then occurs the following inequality ,

$$\left(\frac{a}{b+c}+\frac{b}{c+a}\right)^p+\left(\frac{b}{c+a}+\frac{c}{a+b}\right)^p+\left(\frac{c}{a+b}+\frac{a}{b+c}\right)^p\geq 3\ . \qquad (10)$$

Proof

With the *AM-GM inequality* and then with the inequality from *Proposition* 2 , we have :

$$\sum_{cycl}\left(\frac{a}{b+c}+\frac{b}{c+a}\right)^p\overset{(AM-GM)}{\geq}3\cdot\sqrt[3]{\prod_{cycl}\left(\frac{a}{b+c}+\frac{b}{c+a}\right)^p}=$$

$$=3\cdot\left(\sqrt[3]{\prod_{cycl}\left(\frac{a}{b+c}+\frac{b}{c+a}\right)}\right)^p\overset{(Prop.2)}{\geq}3\ .$$

Equality occurs iff $a=b=c$ or a,b,c arbitrary in $\mathbb{R}_{>0}$ and $p=0$.

8. *Remark*

For $p=1$ we obtain *Nesbitt's inequality* .

The following proposition gives an inequality for exponentials of *Nesbitt lacunary sums*,

9. *Proposition* , [8]

If $a,b,c\in\mathbb{R}_{>0}$ are sides of a triangle , then holds the following inequality ,

$$\left(\frac{a}{b+c}+\frac{b}{c+a}\right)^{\frac{c}{a+b}}+\left(\frac{b}{c+a}+\frac{c}{a+b}\right)^{\frac{a}{b+c}}+\left(\frac{c}{a+b}+\frac{a}{b+c}\right)^{\frac{b}{c+a}}>2 \qquad (11)$$

Proof

For the demonstration we need the following two clarifications :

10. ***Remark***

If a, b, c are the lengths of the sides of a triangle , then ,

$$\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b} \in (0,1) \cdot \qquad (12)$$

Indeed, from $b+c>a$, result , $\frac{a}{b+c}<1$. Analogous : $\frac{b}{c+a}<1$, $\frac{c}{a+b}<1$.

11. ***Lemma*** **(of *Dorin Marghidanu*)** - see for example, [1] , [2]

For $u>0$, $v \in (0,1)$, the following inequality holds :

$$u^v > \frac{u}{u+v} \cdot \qquad (13)$$

(In [1] , there are three proofs of this inequality) .

With these two results we have :

$$\left(\frac{a}{b+c}+\frac{b}{c+a}\right)^{\frac{c}{a+b}}+\left(\frac{b}{c+a}+\frac{c}{a+b}\right)^{\frac{a}{b+c}}+\left(\frac{c}{a+b}+\frac{a}{b+c}\right)^{\frac{b}{c+a}} \overset{Lemma\,11}{>}$$

$$\overset{Lemma\,11}{>} \frac{\frac{a}{b+c}+\frac{b}{c+a}}{\mathcal{N}(a,b,c)}+\frac{\frac{b}{c+a}+\frac{c}{a+b}}{\mathcal{N}(a,b,c)}+\frac{\frac{c}{a+b}+\frac{a}{b+c}}{\mathcal{N}(a,b,c)}=2 \cdot$$

The inequality in Proposition 7 invites the search for similar situations, possible extensions or generalizations .

By an approach similar to that in [3] we will be able to obtain interesting inequalities for entire classes of functions applied to *Nesbitt lacunary sums* , but under certain imposed conditions , thanks to the following ,

12. ***Proposition***

If the function $f: I \subset (0,\infty) \longrightarrow \mathbb{R}_+$ is a *convex* and *increasing* function on I - interval , and so that $1 \in I$, then :

$$f\left(\frac{a}{b+c}+\frac{b}{c+a}\right)+f\left(\frac{b}{c+a}+\frac{c}{a+b}\right)+f\left(\frac{c}{a+b}+\frac{a}{b+c}\right) \geq 3 \cdot f(1) \ , \qquad (14)$$

for any $a, b, c > 0$.

Proof

Indeed, using *Jensen's inequality* for *convex functions* in the first instance, then by

Nesbitt's inequality and also taking into account the fact that the function is *increasing* , we obtain successively :

$$f\left(\frac{a}{b+c}+\frac{b}{c+a}\right)+f\left(\frac{b}{c+a}+\frac{c}{a+b}\right)+f\left(\frac{c}{a+b}+\frac{a}{b+c}\right) \underset{\substack{f\ convexa\\ f(\nearrow)}}{\overset{(Jensen)}{\geq}}$$

$$\underset{\substack{f\ convexa\\ f(\nearrow)}}{\overset{(Jensen)}{\geq}} 3\cdot f\left(\frac{\left(\frac{a}{b+c}+\frac{b}{c+a}\right)+\left(\frac{b}{c+a}+\frac{c}{a+b}\right)+\left(\frac{c}{a+b}+\frac{a}{b+c}\right)}{3}\right)=$$

$$=3\cdot f\left(\frac{2\cdot\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)}{3}\right) \underset{f(\nearrow)}{\overset{(Nesbitt)}{\geq}} 3\cdot f\left(\frac{2\cdot\frac{3}{2}}{3}\right)=3\cdot f(1)\ .$$

Equality occurs if $a=b=c$.

13. ***Remark***

If it is considered $f:(0,\infty)\longrightarrow\mathbb{R}_+$, $f(x)=x^p$, $p\in\mathbb{R}_{\geq 1}$ which is a *convex* and *increasing function* on $(0,\infty)$, we obtain the result from *Proposition* 7 (but slightly weaker , namely only for $p\geqslant 1$) .

14. ***Corollary***

For any $q\geqslant 1$ and for any $a,b,c>0$, the following inequality occurs ,

$$q^{\frac{a}{b+c}+\frac{b}{c+a}}+q^{\frac{b}{c+a}+\frac{c}{a+b}}+q^{\frac{c}{a+b}+\frac{a}{b+c}}\geq 3\cdot q\ , \qquad (15)$$

Proof

Let the function , $f:(0,\infty)\longrightarrow\mathbb{R}_+$, $f(x)=q^x$, $q\geqslant 1$, which is obviously *convex* and *ascending* on $(0,\infty)$, so with *Proposition* 9, we have :

$q^{\frac{a}{b+c}+\frac{b}{c+a}}+q^{\frac{b}{c+a}+\frac{c}{a+b}}+q^{\frac{c}{a+b}+\frac{a}{b+c}}\geq 3\cdot f(1)=3\cdot q$, with equally if $a=b=c$.

15. ***Corollary***

For $a,b,c>0$, there is the inequality ,

$$\sqrt{1+\left(\frac{a}{b+c}+\frac{b}{c+a}\right)^2}+\sqrt{1+\left(\frac{b}{c+a}+\frac{c}{a+b}\right)^2}+\sqrt{1+\left(\frac{c}{a+b}+\frac{a}{b+c}\right)^2}\geq 3\sqrt{2}\ . \qquad (16)$$

Proof

Let the function , $f:I\subset(0,\infty)\longrightarrow\mathbb{R}_+$, $f(x)=\sqrt{1+x^2}$, for which we have :

$f'(x)=\frac{x}{\sqrt{x^2+1}}>0$, $f''(x)=\frac{1}{(x^2+1)\cdot\sqrt{x^2+1}}>0$, so the function f is *convex* and *increasing* on $(0,\infty)$. With *Proposition* 12 , the inequality in the statement follows immediately.

A somewhat dual inequality to that of *Proposition* 12 is provided by

16. *Proposition*

If the function $f:I\subset(0,\infty)\longrightarrow\mathbb{R}_+$ is a *concave* and *descending* function on interval I , then :

$$f\left(\frac{a}{b+c}+\frac{b}{c+a}\right)+f\left(\frac{b}{c+a}+\frac{c}{a+b}\right)+f\left(\frac{c}{a+b}+\frac{a}{b+c}\right)\le 3\cdot f(1)\ , \qquad (17)$$

for any $a,b,c>0$.

Proof

First , using *Jensen inequality* for *concave* functions , then *Nesbitt inequality* and taking into account and the fact that the function is decreasing , we obtain successively :

$$f\left(\frac{a}{b+c}+\frac{b}{c+a}\right)+f\left(\frac{b}{c+a}+\frac{c}{a+b}\right)+f\left(\frac{c}{a+b}+\frac{a}{b+c}\right)\overset{\substack{(Jensen)\\ f\ concava\\ f(\searrow)}}{\le}$$

$$\overset{\substack{(Jensen)\\ f\ concava\\ f(\searrow)}}{\le} 3\cdot f\left(\frac{\left(\frac{a}{b+c}+\frac{b}{c+a}\right)+\left(\frac{b}{c+a}+\frac{c}{a+b}\right)+\left(\frac{c}{a+b}+\frac{a}{b+c}\right)}{3}\right)=$$

$$=3\cdot f\left(\frac{2\cdot\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)}{3}\right)\overset{\substack{(Nesbitt)\\ f(\searrow)}}{\le} 3\cdot f\left(2\cdot\frac{3}{2}\right)=3\cdot f(1)\ .$$

Equality occurs if $a=b=c$.

We also offer here an application for obtaining an inequality, this time using the case of *concave* and *decreasing functions* .

17. *Corollary*

For $a,b,c>0$, there is the inequality ,

$$\sum_{cycl}\left(\frac{a}{b+c}+\frac{b}{c+a}\right)^2+2\cdot\sum_{cycl}\left(\frac{a}{b+c}+\frac{b}{c+a}\right)\ge 9\ . \qquad (18)$$

Proof

Let the function , $f:I\subset(0,\infty)\longrightarrow\mathbb{R}_+$, $f(x)=-x^2-2x+3$, which obviously is a *concave* and *decreasing function* on $(0,\infty)$. Applying *Proposition* 16 , we obtain ,

$$\sum_{cycl}\left[-\left(\frac{a}{b+c}+\frac{b}{c+a}\right)^2-2\cdot\left(\frac{a}{b+c}+\frac{b}{c+a}\right)+3\right]\leq 3\cdot f(1)=0\ ,$$

from which the inequality in the statement immediately follows.

Moreover, the demonstration of the inequality in the statement can also be obtained by applying *Proposition* 7 twice , for the values $p=2$ and $p=1$.

18. ***Remark***

Note that only the possibilities of association : (f – *convex* , f – *ascending*) - from *Proposition* 12 , and (f – *concave* , f – *descending*) - from *Proposition* 16 can be considered . The other two possibilities of association do not ensure the transitivity of the inequality relationship .

Of course – in addition to the applications presented here , numerous other applications can be obtained and demonstrated , following the same scenario .

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