ROMANIAN MATHEMATICAL MAGAZINE

UP.584 Let $a, b, c, d > 1$ and $f: [a, b] \to [c, d]$ a continuous function for which $\exists \lambda \in (a, b)$ such that

$$a \int_a^\lambda f(x)\,dx + b \int_\lambda^b f(x)\,dx \geq a + c$$

then prove:

$$\int_a^b \frac{x}{f(x)}\,dx \leq \left(\frac{1}{a} + \frac{1}{b}\right) \frac{b^2 - a^2 - 2}{2}$$

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Solution by the proposers

Let $F: [a, b] \to \mathbb{R}, F(t) = \int_a^t f(x)\,dx$. Because f is a continuous function, the function F is derivable and $F'(t) = f(t), \forall t \in [a, b]$ we have:

$$\int_a^b xf(x)\,dx \overset{IBP}{=} xF(x)|_a^b - \int_a^b F(x)\,dx = b \int_a^b f(x)\,dx - \int_a^b F(x)\,dx$$

Using Mean Value Theorem:

$$\exists \lambda \in (a, b) \text{ such that } \int_a^b F(x)\,dx = (b - a)F(\lambda) \text{ or}$$

$$\int_a^b xf(x)\,dx = b \int_a^b f(x)\,dx - (b - a)F(\lambda) =$$

$$= b\left(\int_a^\lambda f(x)\,dx + \int_\lambda^b f(x)\,dx\right) - (b - a) \int_a^\lambda f(x)\,dx =$$

$$= a \int_a^\lambda f(x)\,dx + b \int_\lambda^b f(x)\,dx$$

Therefore,

$$a \int_a^\lambda f(x)\,dx + b \int_\lambda^b f(x)\,dx = \int_a^b xf(x)dx \geq a + c \qquad (1)$$

On the other hands, we have:

$$\frac{(c - f(x))(ax - xf(x))}{f(x)} \leq 0, \forall x \in [a, b] \Leftrightarrow$$

$$\frac{acx}{f(x)} - (cx + ax) + xf(x) \leq 0 \Leftrightarrow \frac{x}{f(x)} \leq \left(\frac{1}{a} + \frac{1}{b}\right)x - \frac{1}{ac}xf(x) \Leftrightarrow$$

$$\int_a^b \frac{x}{f(x)}dx \le \left(\frac{1}{a}+\frac{1}{c}\right)\int_a^b x\,dx - \frac{1}{ac}\int_a^b xf(x)\,dx \overset{(1)}{\Leftrightarrow}$$

$$\int_a^b \frac{x}{f(x)}dx \le \left(\frac{1}{a}+\frac{1}{c}\right)\frac{b^2-a^2}{2} - \left(\frac{1}{a}+\frac{1}{c}\right) \Leftrightarrow \int_a^b \frac{x}{f(x)}dx \le \left(\frac{1}{a}+\frac{1}{c}\right)\frac{b^2-a^2-2}{2}$$